9. Cones

Some simple notation: if

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle$$

is a simplex and $w, v_0, v_1, \dots, v_m$ span an $(m+1)$-simplex then

$$\langle w, \sigma \rangle = \langle w, v_0, v_1, \dots, v_m \rangle.$$

Definition-Lemma 9.1. *Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^n$ and let $w \in \mathbb{R}^n$ be a point such that any line through w meets $|\mathcal{K}|$ in at most one point.*

*The **cone** of $\mathcal{K}$ with vertex w, denoted $\operatorname{Cone}_w(\mathcal{K})$, is the simplicial complex with simplices*

$$\langle w, \sigma \rangle$$

and all their faces, where σ is a simplex in $\mathcal{K}$.

If u is any other point with the property that any line through u meets $|\mathcal{K}|$ in at most one point then $\operatorname{Cone}_w(\mathcal{K})$ and $\operatorname{Cone}_u(\mathcal{K})$ are isomorphic simplicial complexes.

Proof. We first check that $\operatorname{Cone}_w(\mathcal{K})$ is a simplicial complex. Note that w cannot lie in the affine subspace spanned by σ. It follows that $w, v_0, v_1, \dots, v_m$ span an $(m+1)$-simplex.

Simplices in $\operatorname{Cone}_w(\mathcal{K})$ come in three types

(1) $\langle w, \sigma \rangle$, where σ is a simplex in $\mathcal{K}$,
(2) σ, where σ is a simplex in $\mathcal{K}$, and
(3) w.

In particular it is clear that if $\sigma \in \operatorname{Cone}_w(\mathcal{K})$ and τ is a face of σ then $\tau \in \operatorname{Cone}_w(\mathcal{K})$.

Now suppose that σ and τ are two simplices in $\operatorname{Cone}_w(\mathcal{K})$. Suppose $\rho = \sigma \cap \tau$ is non-empty. If σ and τ are simplices in $\mathcal{K}$ then ρ is certainly a face of both σ and τ.

If $\sigma = \langle w, \sigma_0 \rangle$, where σ_0 and τ are simplices in $\mathcal{K}$ then

$$\rho = \sigma_0 \cap \tau.$$

Then ρ is a face of both σ_0 and τ, so that ρ is also a face of σ.

Similarly if

$$\sigma = \langle w, \sigma_0 \rangle \qquad \text{and} \qquad \tau = \langle w, \tau_0 \rangle$$

where σ_0 and τ_0 are simplices in $\mathcal{K}$ then

$$\rho_0 = \sigma_0 \cap \tau_0,$$

is a face of both σ_0 and τ_0, so that

$$\rho = \langle w, \rho_0 \rangle$$

is a face of σ and τ. It follows that $\mathrm{Cone}_w(\mathcal{K})$ is a simplicial complex.

It is clear that the abstractions of $\mathrm{Cone}_w(\mathcal{K})$ and $\mathrm{Cone}_u(\mathcal{K})$ are isomorphic abstract simplicial complexes, so that $\mathrm{Cone}_w(\mathcal{K})$ and $\mathrm{Cone}_u(\mathcal{K})$ are isomorphic simplicial complexes. $\square$

Definition-Lemma 9.2. *A **chain homotopy** between two simplicial maps f and g of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ is a sequence of group homomorphisms*

$$P_m \colon C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{L})$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f_* - g_*.$$

If f and g are chain homotopic then

$$f_* = g_* \colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}).$$

Proof. Suppose that α is an m-cycle. Then

$$\begin{aligned} f_*\alpha - g_*\alpha &= (f_* - g_*)\alpha \\ &= (\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)\alpha \\ &= (\partial_{m+1} \circ P_m)\alpha + (P_{m-1} \circ \partial_m)\alpha \\ &= \partial_{m+1} P_m(\alpha) + P_{m-1}(\partial_m \alpha) \\ &= \partial_{m+1}\beta, \end{aligned}$$

where $\beta = P_m(\alpha) \in C_{m+1}(\mathcal{L})$. Thus

$$f_*\alpha \qquad \text{and} \qquad g_*\alpha$$

are homologous, that is, they are equal in the homology group $H_m(\mathcal{L})$, as their difference is a boundary. $\square$

The following diagram

$$\begin{tikzcd} C_{m+1}(\mathcal{K}) \arrow[r, "\partial_{m+1}"] \arrow[d, "\pi"'] & C_m(\mathcal{K}) \arrow[r, "\partial_m"] \arrow[d, "\pi"'] \arrow[dl, "P_m"'] & C_{m-1}(\mathcal{K}) \arrow[d, "\pi"'] \arrow[dl, "P_{m-1}"'] \\ C_{m+1}(\mathcal{L}) \arrow[r, "\partial_{m+1}"] & C_m(\mathcal{L}) \arrow[r, "\partial_m"] & C_{m-1}(\mathcal{L}) \end{tikzcd}$$

where $\pi = f_* - g_*$ might be enlightening.

Theorem 9.3. *If $\mathcal{L} = \mathrm{Cone}_w(\mathcal{K})$ is the cone over a simplicial complex $\mathcal{K}$ then the reduced cohomology of $\mathcal{L}$ is zero,*

$$\tilde{H}_m(\mathcal{L}) = 0 \qquad \textit{for all} \qquad m.$$

Proof. $\mathcal{L}$ is connected and so

$$\tilde{H}_0(\mathcal{L}) = 0.$$

Now suppose $m > 0$. Define a group homomorphism

$$P\colon C_m(\mathcal{L}) \longrightarrow C_{m+1}(\mathcal{L})$$

as follows. On simplices σ one defines P by the rule

$$P(\sigma) = \begin{cases} \langle w, \sigma \rangle & \text{if } w \text{ is not a vertex of } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then one extends P by linearity to all m-chains. We will also use the notation

$$P(\alpha) = \langle w, \alpha \rangle.$$

To compute the RHS one expands by linearity. If $\alpha = \sigma$ is a simplex and $w \in \sigma$ then the RHS is zero.

Claim 9.4. *If* $\alpha \in C_m(\mathcal{L})$ *then*

$$\partial_{m+1}\langle w, \alpha \rangle = \alpha - \langle w, \partial_m \alpha \rangle.$$

Proof of (9.4)*.* First suppose that $\alpha = \sigma$ is an m-simplex. We want to show that

$$\partial_{m+1}\langle w, \sigma \rangle = \sigma - \langle w, \partial_m \sigma \rangle.$$

If σ does not involve w then this is a straightforward consequence of the formula for ∂_{m+1}.

If σ involves w then the LHS is zero. To check we have equality there is no harm in assumming that the first element of σ is w. The only term in the expansion of

$$\partial_m \sigma$$

that survives when we tack on w at the front is the one that drops w. Since this is the first term of σ the term

$$\langle w, \partial_m \sigma \rangle = \sigma.$$

Thus the RHS is zero as well.

Now use the fact that both sides are linear in α. □

We have

$$\begin{aligned}
(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\alpha) &= \partial_{m+1} P_m(\alpha) + P_{m-1}(\partial_m \alpha) \\
&= \partial_{m+1}\langle w, \alpha \rangle + \langle w, \partial_m \alpha \rangle \\
&= \alpha - \langle w, \partial_m \alpha \rangle + \langle w, \partial_m \alpha \rangle \\
&= \alpha \\
&= \mathrm{id}_*(\alpha).
\end{aligned}$$

Thus the identity map is chain homotopic to the zero map. It follows that the identity map and the zero map are equal on homology. This is only possible if the homology groups are zero. $\square$