7. The pushforward

We will need a simple:

Definition 7.1. *A simplicial map f between two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ is a function*

$$f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

with the property that if $v_0, v_1, \dots, v_m$ are vertices of a simplex $\sigma \in \mathcal{K}$ then $w_0, w_1, \dots, w_m$ are vertices of a simplex $\tau \in \mathcal{L}$, where $w_i = f(v_i)$ and the restriction of f to σ is linear.

Here are some simple observations about simplicial maps. Note first that we don't require that τ has the same dimension as σ, that is, we allow repetitions in the sequence $w_0, w_1, \dots, w_m$. Note also that f is determined by what it does to the vertices. In fact f is simply the map constructed in Question 6, part (iii) of Homework 1.

Observe that simplicial maps are always continuous (since linear functions are continuous). Note also that one can compose simplicial maps. If

$$f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}| \qquad \text{and} \qquad g\colon |\mathcal{L}| \longrightarrow |\mathcal{M}|$$

are two simplicial maps then let

$$h = g \circ f\colon |\mathcal{K}| \longrightarrow |\mathcal{M}|$$

be the composition, as functions.

If σ is a simplex then the image of σ inside $|\mathcal{L}|$ is a simplex τ in $\mathcal{L}$. The image of τ in $|\mathcal{M}|$ is a simplex ρ in $\mathcal{M}$. Thus the image σ in $|\mathcal{M}|$ is the same simplex ρ in $\mathcal{M}$. The restriction of f to σ is linear and the restriction of g to τ is linear, so that the restriction of h to σ is linear, as the composition of linear maps is linear. Thus h is a simplicial map and we can compose simplicial maps (we could have also used the notion of abstract simplicial complex introduced in Question 6 of Homework 1 to show this).

Thus we get a category, whose objects are simplicial complexes and whose morphisms are simplicial maps.

Now we show how we can use simplicial maps to compare the homology of $\mathcal{K}$ and $\mathcal{L}$. We want to compare chain complexes. The basic observation is that the definition of boundary maps does not really depend on the chain complex (once we have chosen an ordering of the vertices).

Note first that we can order the vertices of $\mathcal{K}$ and then order the vertices of $\mathcal{L}$ so that f preserves this order (this is easy. f induces an ordering of the vertices in the image of f and we then just arbitrarily extend this to an ordering of the vertices of $\mathcal{L}$).

We first want to define a group homomorphism

$$f_*\colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}).$$

It is enough to define the image of a m-simplex and then extend linearly.

Definition 7.2. *Suppose we are given a simplicial map $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.*

Then we get a sequence of group homomorphisms

$$f_*\colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}).$$

If σ is an m-simplex then let τ be the image of f. We define

$$f_*\sigma = \begin{cases} \tau & \text{if } \tau \text{ is an } m\text{-simplex} \\ 0 & \text{otherwise.} \end{cases}$$

We then extend f_* linearly to all m-chains.

Note a key aspect of this definition. If the image of σ has lower dimension then the pushforward is zero.

In terms of notation, note that there should really be an index on f_*. If it turns out to be really necessary to keep track of this, we will actually use the notation $C_m(f)$.

Lemma 7.3. *Suppose we are given a simplicial map $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.*

Then the following diagram commutes

$$\begin{array}{ccc} C_m(\mathcal{K}) & \xrightarrow{\partial_m} & C_{m-1}(\mathcal{K}) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ C_m(\mathcal{L}) & \xrightarrow{\partial_m} & C_{m-1}(\mathcal{L}). \end{array}$$

Proof. We want to check that

$$f_* \circ \partial_m = \partial_m \circ f_*.$$

Since both sides are group homomorphisms, it suffices to check this on m-simplices

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle.$$

Let $w_i = f(v_i)$.

There are two cases.

Suppose that the image τ of σ is an m-simplex. We compute both sides. We start with the LHS:

$$\begin{aligned}(f_* \circ \partial_m)\sigma &= (f_* \circ \partial_m)\langle v_0, v_1, \dots, v_m\rangle \\ &= f_* \sum_i (-1)^i \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m\rangle \\ &= \sum_i (-1)^i f_* \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m\rangle \\ &= \sum_i (-1)^i \langle w_0, w_1, \dots, \hat{w}_i, \dots, w_m\rangle.\end{aligned}$$

Note that since τ is an m-simplex, all of its faces are $(m-1)$-simplices and this gives us the last equality. Now we turn to the RHS

$$\begin{aligned}(\partial_m \circ f_*)\sigma &= \partial_m \langle w_0, w_1, \dots, w_m\rangle \\ &= \sum_i (-1)^i \langle w_0, w_1, \dots, \hat{w}_i, \dots, w_m\rangle,\end{aligned}$$

and the two sides are obviously equal.

Now suppose that τ has lower dimension than σ. Then the RHS is zero and we have to check that the LHS is zero. We have to show

$$\sum_i (-1)^i f_* \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m\rangle = 0.$$

Now the only way that the RHS is zero is that two vertices of τ are repeated. Suppose that $w_i = w_j$. If drop an index other than i or j in the sum above, then we still have a repeated index and the pushforward of this term is by definition zero. Similarly if there are more repeated vertices, then we obviously get zero, regardless of which vertex we drop.

Thus we are reduced to showing that

$$(-1)^i \langle w_0, w_1, \dots, \hat{w}_i, \dots, w_m\rangle + (-1)^j \langle w_0, w_1, \dots, \hat{w}_j, \dots, w_m\rangle = 0.$$

We may assume that $i < j$. Note that

$$w_0, w_1, \dots, \hat{w}_i, \dots, w_m \qquad \text{and} \qquad w_0, w_1, \dots, \hat{w}_j, \dots, w_m,$$

have the same set of vertices, just in a different order. In the first term, the vertex at position $j-1$ (note that we dropped the lower index i) is $w_j = w_i$ and in the second term, the vertex at position i is $w_i = w_j$. If we want to move the w_i at position i to the position $j-1$ then we need to make $j-i-1$ swaps. Thus

$$\langle w_0, w_1, \dots, \hat{w}_i, \dots, w_m\rangle = (-1)^{j-i-1} \langle w_0, w_1, \dots, \hat{w}_j, \dots, w_m\rangle.$$

It follows that the RHS is:

$$\begin{aligned}(f_* \circ \partial_m)\sigma &= \sum_i (-1)^i f_* \langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m \rangle \\ &= (-1)^i \langle w_0, w_1, \dots, \hat{w}_i, \dots, w_m \rangle + (-1)^j \langle w_0, w_1, \dots, \hat{w}_j, \dots, w_m \rangle \\ &= (-1)^{j-1} \langle w_0, w_1, \dots, \hat{w}_j, \dots, w_m \rangle + (-1)^j \langle w_0, w_1, \dots, \hat{w}_j, \dots, w_m \rangle \\ &= 0. \qquad \square\end{aligned}$$

Example 7.4. *Let us look as a simple example to see how the proof works.*

Suppose that

$$\sigma = \langle v_0, v_1, v_2, v_3, v_4 \rangle$$

is a 4-simplex. Suppose that $f(v_i) = w_i$, and that $w_1 = w_4$ is the only repeated vertex. Then

$$f_*\sigma = 0,$$

due to the repeated vertex.

If we compute the LHS we are down to computing

$$f_*\langle v_1, v_2, v_3, v_4 \rangle - f_*\langle v_0, v_2, v_3, v_4 \rangle + f_*\langle v_0, v_1, v_3, v_4 \rangle - f_*\langle v_0, v_1, v_2, v_4 \rangle + f_*\langle v_0, v_1, v_2, v_3 \rangle.$$

The first, third and fourth terms are zero, since the image of the corresponding vertices has a repeated term. This leaves

$$-f_*\langle v_0, v_2, v_3, v_4 \rangle + f_*\langle v_0, v_1, v_2, v_3 \rangle = -\langle w_0, w_2, w_3, w_4 \rangle - \langle w_0, w_1, w_2, w_3 \rangle.$$

Since $w_1 = w_4$ this is the same as

$$-\langle w_0, w_2, w_3, w_1 \rangle + \langle w_0, w_1, w_2, w_3 \rangle.$$

We try to make the first expansion look like the second. Note that swapping the second and third terms (starting the count at zero, as in the proof) and then the first and second terms, we get

$$\begin{aligned}\langle w_0, w_2, w_3, w_1 \rangle &= -\langle w_0, w_2, w_1, w_3 \rangle \\ &= \langle w_0, w_1, w_2, w_3 \rangle.\end{aligned}$$

Thus

$$-\langle w_0, w_1, w_2, w_3 \rangle + \langle w_0, w_1, w_2, w_3 \rangle = 0.$$

Lemma 7.5. *Suppose we are given a simplicial map $f \colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.*

Let $\alpha \in C_m(\mathcal{K})$ be an m-chain and let $\gamma = f_*\alpha \in C_m(\mathcal{K})$ be the pushforward of α.

1. *If α an m-cycle then γ is an m-cycle.*
2. *If α an m-boundary then γ is an m-boundary.*

Proof. Suppose that α is an m-cycle. We have

$$\begin{aligned}\partial_m\gamma &= \partial_m(f_*\alpha)\\ &= (\partial_m \circ f_*)\alpha\\ &= (f_* \circ \partial_m)\alpha\\ &= f_*(\partial_m\alpha)\\ &= f_*0\\ &= 0.\end{aligned}$$

Thus γ is an m-cycle.

Now suppose that α is an m-boundary. Then we may find an $(m+1)$-chain $\beta \in C_{m+1}(\mathcal{K})$ such that $\partial_{m+1}\beta = \alpha$. Let $\delta = f_*\beta \in C_{m+1}(\mathcal{K})$. We have

$$\begin{aligned}\partial_{m+1}\delta &= \partial_{m+1}(f_*\beta)\\ &= (\partial_{m+1} \circ f_*)\beta\\ &= (f_* \circ \partial_{m+1})\beta\\ &= f_*(\partial_{m+1}\beta)\\ &= f_*\alpha\\ &= \gamma.\end{aligned}$$

Thus γ is an m-boundary. □

In words, the pushforward respects boundaries and cycles.

Definition-Lemma 7.6. *Suppose we are given a simplicial map $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.*

Then we get a sequence of group homomorphisms

$$f_*\colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}).$$

Proof. We have already seen that there is a pushforward map on chains

$$f_*\colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}).$$

If we compose f_* with the natural inclusion of m-cycles into m-chains,

$$Z_m(\mathcal{K}) \longrightarrow C_m(\mathcal{K})$$

then (7.5) implies that we get a pushforward map on m-cycles

$$f_*\colon Z_m(\mathcal{K}) \longrightarrow Z_m(\mathcal{L}).$$

Suppose that we compose this with the natural quotient map

$$Z_m(\mathcal{L}) \longrightarrow H_m(\mathcal{L}).$$

This gives a map

$$\phi\colon Z_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}).$$

Now the kernel of the quotient map is the subgroup of m-boundaries of $\mathcal{L}$. (7.5) implies that the m-boundaries of $\mathcal{K}$ are sent to the m-boundaries of $\mathcal{L}$, so that the kernel of ϕ contains the m-boundaries of $\mathcal{K}$. But then the universal property of the quotient implies that there is a natural induced group homomorphism

$$f_*\colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}). \qquad \square$$

Example 7.7. *Let us take a triangulation $\mathcal{K}$ of a torus, induced from a triangulation of I^2.*

Pick one with nine squares and eighteen triangles. I^2 maps down to I via projection. If we divide I into three equal pieces, the induced map on each 2-simplex is a linear map down to the corresponding 1-simplex.

This induces a simplicial map

$$f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

where $\mathcal{L}$ is the induced triangulation of S^1.

Consider the induced map on homology. The image of any 2-simplex is a 1-simplex. Thus

$$f_*\colon H_2(\mathcal{K}) \longrightarrow H_2(\mathcal{L})$$

is the zero map. Of course the same is true for all higher homology groups, since these all vanish. The image of a vertex is a vertex. As both the torus and S^1 are connected it follows that

$$f_*\colon H_0(\mathcal{K}) \longrightarrow H_0(\mathcal{L})$$

is an isomorphism. Indeed both groups are isomorphic to $\mathbb{Z}$ and the degree of a 0-cycle in $\mathcal{K}$ is the same as the degree of the pushforward.

We turn to the first homology. Recall that

$$H_1(\mathcal{L}) \simeq \mathbb{Z}$$

with generator

$$\gamma = \langle w_0, w_1 \rangle + \langle w_1, w_2 \rangle + \langle w_2, w_0 \rangle.$$

This is an easy computation, since $\partial_1 \gamma = 0$ and there are no 2-simplices to kill γ.

Let the vertices along the top edge of I^2 be v_0, v_1, v_2. Then

$$\alpha = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle$$

is a 1-cycle. If we apply f_* then we get γ. As γ is non-zero it follows that α is not homologous to zero.

In fact, considering the projection to both factors, f and g, we can then show that no combination of α and a similar vertical cycle β is zero.

If we start with

$$a\alpha + b\beta$$

then

$$f_*(a\alpha + b\beta) = a\gamma \qquad \text{and} \qquad g_*(a\alpha + b\beta) = b\gamma.$$

If follows that if $a\alpha + b\beta$ is homologous to zero then $a = b = 0$.

Lemma 7.8. *Suppose we are given two simplicial maps $f\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$ and $g\colon |\mathcal{L}| \longrightarrow |\mathcal{M}|$ of simplicial complexes $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$.*

Then

$$(g \circ f)_* = (g_* \circ f_*)\colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{M}).$$

Proof. We first show that

$$(g \circ f)_* = (g_* \circ f_*)\colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{M}).$$

This is easy. Both sides are group homomorphisms and so we only need to check that we get equality on m-simplices. Suppose that

$$\sigma = \langle v_0, v_1, \dots, v_m\rangle.$$

Let $w_i = f(v_i)$ and $x_i = g(w_i)$. Then $x_i = h(v_i)$. If there are any repetitions in $x_0, x_1, \dots, x_m$ then both sides are zero. Otherwise both sides are

$$\rho = \langle x_0, x_1, \dots, x_m\rangle.$$

It is then clear that both sides give the same map

$$(g \circ f)_* = (g_* \circ f_*)\colon Z_m(\mathcal{K}) \longrightarrow Z_m(\mathcal{M})$$

and it is not much harder to conclude that we get equality on homology. □