6. Reduced homology and stars

It is interesting to try to do some of these computations a little bit more conceptually.

Theorem 6.1. *Let $\mathcal{K}$ be a simplicial complex.*

Then

$$H_0(\mathcal{K}) \simeq \mathbb{Z}^r$$

where r is the number of connected components of $|\mathcal{K}|$.

We will need a bunch of results:

Lemma 6.2. *Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{L}$ be its 1-skeleton.*

Then two vertices v and w belong to the same connected component of $|\mathcal{K}|$ if and only if they belong to the same connected component of $|\mathcal{L}|$.

In particular the connected components of $\mathcal{K}$ are subcomplexes and the path components and connected components of $|\mathcal{K}|$ coincide.

For this we will need:

Definition 6.3. *Let σ be a simplex. The* ***boundary*** *of σ is the union of the proper faces of σ. The* ***interior*** *of σ is the complement of the boundary.*

Note if σ is an m-simplex then the support of the $(m-1)$-skeleton of σ and the boundary are the same.

Notice that the definition of the interior of σ is in conflict with the definition of the interior of a subset of $\mathbb{R}^n$. If the vertices of σ generate the whole of $\mathbb{R}^n$ they coincide. For this reason the interior of a simplex is sometimes called the relative interior. It is the interior in the affine space spanned by the vertices.

Definition-Lemma 6.4. *Let $\mathcal{K}$ be a simplicial complex and let v be a vertex of $\mathcal{K}$.*

The ***star*** *of v, denoted $\operatorname{St}(v)$, is the union of the interiors of the simplices that contain v.*

$\operatorname{St}(v)$ *is an open subset of $|\mathcal{K}|$.*

For this we will need:

Lemma 6.5. *Let $\mathcal{K}$ be a simplicial complex and let v be a vertex of $\mathcal{K}$.*

Define a function

$$\lambda_v \colon |\mathcal{K}| \longrightarrow \mathbb{R}$$

by the rule

$$\lambda_v(x) = \begin{cases} \lambda_v^\sigma(x) & \text{if } x \text{ and } v \text{ belong to the same simplex } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Here $\lambda_v^\sigma(x)$ refers to the v component of the barycentric coordinates of x in σ.

Proof. Note that we already saw that $\lambda_v = \lambda_v^\sigma$ is continuous on each simplex σ to which v belongs. It is certainly continuous on any simplex to which it does not belong, since then it is the zero function.

Let σ and τ be simplices with non-empty intersection ρ. If If σ and τ are two simplices that contain v then we already saw that λ_v agrees on the intersection ρ. If σ and τ do not contain v then it is clear that λ_v agrees on the intersection, since it is the zero function on both σ and τ. If v belongs to σ but not τ then ρ does not contain v, so that

$$\lambda_v^\sigma|_\rho = 0,$$

the zero function.

Note that the simplices of $|\mathcal{K}|$ are closed subsets. Thus λ_v patches to a continuous function. □

Proof of (6.2). Note every point x of $\mathrm{St}(v)$ belongs to the interior of a unique simplex σ with vertex v.

Thus $\lambda(x) = \lambda_v^\sigma(x) > 0$. It follows that the points of $\mathrm{St}(v)$ are the points such that the function $\lambda_v > 0$. In particular $\mathrm{St}(v)$ is open. □

Proof of (6.2). Define an equivalence relation $\sim$ on the vertices V by the rule, $v \sim w$, if there is a sequence

$$v = v_0, \langle v_0, v_1\rangle, v_1, \langle v_1, v_2\rangle, \dots, v_{n-1}, \langle v_{n_1}, v_n\rangle, v_n = w.$$

It is not hard to check that $\sim$ is an equivalence relation. It is clear that we will get the same equivalence relation, whether we work with $|\mathcal{K}|$ or $|\mathcal{L}|$.

Given a vertex v, let

$$U_v = \{\, \mathrm{St}(w) \,|\, w \sim v \,\}.$$

Here we take the star in the complex $\mathcal{K}$. Note that U_v is open as it is a union of open sets.

I claim that U_v is path connected. Indeed, given two points a and b of U_v, they belong to the interior of two simplices. Thus we may find two vertices u and w such that $a \in \mathrm{St}(u)$ and $b \in \mathrm{St}(w)$. Now there is a path from u to W by assumption and there are paths from a to u and b to v (in fact straight lines). Thus there is a path from a to b and so U_v is path connected.

Now suppose that U_v and U_w intersect. Suppose that $x \in U_v \cap U_w$. Then there are two vertices t and u such that $t \sim v$, $u \sim w$ and $x \in \mathrm{St}(t)$, $x \in \mathrm{St}(u)$. Since x belongs to the interior of a unique simplex σ, it follows t and u both belong to σ. In particular $\langle t, u\rangle$ is a 1-simplex and so $t \sim u$. But then $v \sim u$ and so $U_v = U_w$.

Thus U_v are the connected components of $|\mathcal{K}|$. It is clear that we will get a similar answer if we work with $|\mathcal{L}|$ instead of $|\mathcal{K}|$. □

Proof of (6.1). Pick vertices $v_1, v_2, \dots, v_r$ from each connected component of $|\mathcal{K}|$.

Suppose that w is a vertex of $|\mathcal{K}|$. Then $w \sim v_i$, for some i. It follows that there is a sequence of vertices,

$$w = w_1, w_2, \dots, w_k = v_i,$$

such that $\langle w_j, w_{j+1}\rangle$ is a 1-simplex. Consider the 1-chain

$$\beta = \sum_{j=1}^{k} \langle w_j, w_{j+1}\rangle.$$

We have

$$\begin{aligned}
\partial_1\beta &= \partial_1 \sum_{j=1}^{k} \langle w_j, w_{j+1}\rangle \\
&= \sum_{j=1}^{k} \partial_1\langle w_j, w_{j+1}\rangle \\
&= \sum_{j=1}^{k} w_{j+1} - w_j \\
&= v_i - w.
\end{aligned}$$

Thus show that w is homologous to v_i (that is, the class of w is equal to the class of v_i in homology).

It follows that every 0-cycle is homologous to

$$\alpha = \sum_{i=1}^{k} a_i v_i.$$

Next we show that none of these 0-cycles are homologous to zero. Note that each component of $|\mathcal{K}|$ has a degree map,

$$d_i \colon C_0(\mathcal{K}) \longrightarrow \mathbb{Z}$$

d_i sends a 0-chain to the sum of all of the coefficients of all the vertices that belong to that component. It is easy to see that the boundary of any 1-simplex has degree zero. Indeed, either one of the vertices belongs to another component, in which case both vertices belong to another component, and the degree is obviously zero, or d_i is zero, since both vertices belong to this component and $1 - 1 = 0$. But then the boundaries lie in the kernel of d_i.

On the other hand

$$d_i(\alpha) = a_i,$$

so that if α is a boundary, then $\alpha = 0$. It follows that the map

$$H_0(\mathcal{K}) \longrightarrow \mathbb{Z}^r$$

which sends α to the r-tuple

$$(a_1, a_2, \ldots, a_r)$$

is an isomorphism. □

Note that the zeroth homology group is never equal to zero. We can exploit this fact and change the chain complex to get the reduced homology.

The **augmented chain complex** is the same as the usual chain complex, except we that add an extra term on the RHS:

$$\cdots \longrightarrow C_n(\mathcal{K}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{K}) \longrightarrow \cdots \longrightarrow C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\partial_{-1}} 0.$$

The map

$$\epsilon \colon C_0(\mathcal{K}) \longrightarrow \mathbb{Z}$$

is simply the degree map, which sends

$$\sum a_i v_i$$

to the sum

$$\sum a_i.$$

We already saw that ϵ is a group homomorphism and we already checked that

$$\epsilon \circ \partial_1 = 0,$$

since the degree of a boundary is zero. On the other hand it is clear that

$$\partial_{-1} \circ \epsilon = 0,$$

since the last map is the zero map. Thus we do have a chain complex. ϵ is called the **augmentation map**.

Definition-Lemma 6.6. *Let $\mathcal{K}$ be a simplicial complex.*

*The **reduced homology** of $\mathcal{K}$, denoted $\tilde{H}_i(\mathcal{K})$, is the homology of the complex above.*

We have

$$H_i(\mathcal{K}) = \begin{cases} \tilde{H}_0(\mathcal{K}) \oplus \mathbb{Z} & \text{if } i = 0 \\ \tilde{H}_i(\mathcal{K}) & \text{otherwise.} \end{cases}$$

In particular $\tilde{H}_0(\mathcal{K})$ *is free abelian of rank one less than the number of connected components of* $|\mathcal{K}|$.

Proof. It is clear that the chain complex and the augmented chain complex agree except in degree 0 and lower.

Thus the last equality is clear.

Pick vertices $v_0, v_1, \dots, v_{r-1}$ from each connected component of $|\mathcal{K}|$. We already saw that every 0-cycle is uniquely homologous to a sum $\sum a_i v_i$.

Note that $v_i - v_0$ is a 0-chain that is sent to zero by ϵ. None of these are boundaries. Define a map

$$\phi\colon H_0(\mathcal{K}) \longrightarrow \tilde{H}_0(\mathcal{K}) \oplus \mathbb{Z}$$

by sending

$$\sum a_i v_i \qquad \text{to} \qquad (\sum a_i(v_i - v_0), \sum a_i).$$

Then ϕ is a group homomorphism. It is easy to see that ϕ is a bijection. □

It seems worth pointing out that the isomorphism we constructed is not natural. It depended on the choice of v_0, or better, on the choice of connected component containing v_0.