5. Some computations

Let us compute in some easy examples.

Example 5.1. *Let $\mathcal{K}$ be the 1-skeleton of a 2-simplex.*

Note that $\mathcal{K}$ defines a triangulation of the sphere S^1. We compute its homology groups.

Let the vertices of $\mathcal{K}$ be v_0, v_1 and v_2. Then there are three edges,

$$\sigma_0 = \langle v_1, v_2 \rangle, \qquad \sigma_1 = \langle v_0, v_2 \rangle \qquad \text{and} \qquad \sigma_2 = \langle v_0, v_1 \rangle.$$

Thus

$$C_0(\mathcal{K}) \simeq \mathbb{Z}^3 \qquad \text{and} \qquad C_1(\mathcal{K}) \simeq \mathbb{Z}^3.$$

On the other hand, $C_n(\mathcal{K}) = 0$ for all $n > 1$. The relevant part of the chain complex is then

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 \longrightarrow 0.$$

Two easy, but very useful observations. Everything in

$$C_0(\mathcal{K}) = Z_0(\mathcal{K})$$

is a cycle, since ∂_0 is the zero map. Nothing in $C_1(\mathcal{K})$ is a boundary, since $C_2(\mathcal{K}) = 0$. Thus $B_1(\mathcal{K}) = 0$.

Thus $H_1(\mathcal{K})$ is the kernel of ∂_1 and $H_0(\mathcal{K})$ is the cokernel of ∂_1 (which just means quotient out by the image of ∂_1).

Note that

$$\partial_1\sigma_0 = v_2 - v_1, \qquad \partial_1\sigma_1 = v_2 - v_0 \qquad \text{and} \qquad \partial_1\sigma_2 = v_1 - v_0.$$

This is efficiently encoded in a 3×3 matrix with integer coefficients

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We want to know the kernel and image (or better cokernel) of this matrix. This is almost the same as calculating the kernel and image as a vector space but one has to be careful with torsion. For example, consider the following sequence of examples

$$f\colon \mathbb{Z} \longrightarrow \mathbb{Z}.$$

f is determined by the image of 1. If this is n then the cokernel is

$$\frac{\mathbb{Z}}{n\mathbb{Z}}.$$

So the terms are the same but the cokernel depends heavily on the map f. What can change is the torsion.

There is a general algorithm to compute the kernel and the image. The idea is to follow the same steps as Gaussian elimination. To keep track of torsion, we simply don't divide. We first swap the first and third rows:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Now add the first row to the second

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Now we add the second row to the third

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The kernel is a copy of $\mathbb{Z}$. A generator for the kernel is

$$\sigma_0 - \sigma_1 + \sigma_2.$$

The image is a copy of $\mathbb{Z}^2$. Since the entries on the main diagonal are 1, 1 and 0, the quotient is a copy of $\mathbb{Z}$. Thus

$$H_0(\mathcal{K}) = \mathbb{Z} \qquad \text{and} \qquad H_1(\mathcal{K}) = \mathbb{Z}.$$

We can do these computations a little bit more conceptually. Note that if we start with a 1-simplex, then the boundary is the difference of the two endpoints. Thus any two vertices are equivalent. On the hand every 0-cycle is the sum of the vertices

$$a_0 v_0 + a_1 v_1 + a_2 v_2.$$

Any such has a natural degree

$$a_0 + a_1 + a_2.$$

The assignment

$$Z_0(\mathcal{K}) \longrightarrow \mathbb{Z}$$

which sends a 0-cycle to its degree is a group homomorphism. Any boundary has degree zero, since the boundary of a 1-simplex has degree zero. In fact the kernel of the degree map is simply the set of boundaries.

Thus

$$H_0(\mathcal{K}) \simeq \mathbb{Z}$$

by the first isomorphism theorem.

What about the 1-cycles? The fact that the image of a 1-cycle is zero exactly says that the 1-cycle contains the same number of ingoing and outgoing 1-simplices at a vertex.

Recall the notion of a cycle, from graph theory. This is a sequence of vertices and edges

$$a_0, a_0a_1, a_1, a_1a_2, \dots, a_na_0, a_0,$$

where the only repeated vertex is a_0.

In our case there is clearly only one cycle in the sense of graph theory and this generates $H_1(\mathcal{K})$.

Example 5.2. *Now let $\mathcal{K}$ be the simplicial complex determined by a 2-simplex.*

Note that we have not changed anything except we added one 2-simplex σ. Thus the relevant part of the chain complex becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 \longrightarrow 0.$$

Note that

$$C_2(\mathcal{K}) = \mathbb{Z}$$

is generated by σ.

$$\partial\sigma = \sigma_0 - \sigma_1 + \sigma_2.$$

Thus the generator of the 1-cycles is also a boundary. It follows that

$$H_0(\mathcal{K}) = \mathbb{Z} \qquad \text{and} \qquad H_1(\mathcal{K}) = 0.$$

The fact that the first homology is non-zero in the first example and zero in the second example points to the fact that in the first example there is a hole but in the second example the hole disappeared after we added a 2-simplex.

One more computation:

Example 5.3. *We give the annulus a triangulation $\mathcal{K}$ as follows.*

Start with the triangulation of the unit square using three vertical lines and one horizontal line, and then adding six diagonals. This induces a triangulation $\mathcal{K}$ of the annulus.

We first compute $H_0(\mathcal{K})$. Note that the image of any two vertices is equivalent in $H_0(\mathcal{K})$. Indeed, just pick a path from one vertex to the other. Summing over the boundaries of each edge in this path tells us the vertices are equivalent in the quotient $H_0(\mathcal{K})$. On the other hand, the degree of any boundary is zero. Thus the degree map establishes an isomorphism

$$H_0(\mathcal{K}) \longrightarrow \mathbb{Z}.$$

Now consider the 2-cycles. Suppose we could form a cycle from these. Then there would have to be cancelling along the edges. A priori this is possible for one of the internal triangles, since two triangles share internal edges.

But the top three triangles are the only triangles with a horizontal side at the top. So the coefficient of these triangles in any cycle is zero. By the same token the same is true for the bottom three triangles. Now consider a triangle in the middle. It shares a diagonal edge with a triangle at the top or bottom.

But we already decided that the coefficient of any of these triangles in a cycle is zero. Thus the coefficient of any middle triangle in a cycle is also zero. Thus there are no 2-cycles (apart from the trivial zero 2-cycle). It follows that

$$H_2(\mathcal{K}) = 0.$$

Now consider $H_1(\mathcal{K})$. First consider the 1-skeleton. This gives us a graph. Consider a cycle in the graph. If the length of this cycle is longer than 3 then we can write it is a sum of cycles of length 3. There are 15 cycles of length 3. 12 come from the triangles and there are three more coming from the three horizontal lines. Now when we put back in the 12 triangles, this kills the 12 cycles coming from the triangles. The 3 remaining cycles come from the 3 horizontal lines. One can check, after a little bit of computation, that these are all equivalent modulo boundaries. The idea is to keep using the fact that if we have two ways to go around a 2-cycle then they are equivalent.

Thus every 1-cycle is equivalent to a multiple of the 1-cycle given by the top horizontal line. It remains to check this is not zero (or torsion).

There are two ways to show this. The first is by direct computation. The idea is to use the same trick as we used above to show that there are no 2-cycles. If we have a 1-cycle with enough coefficients that are zero then we can show that it cannot be the boundary of a 2-chain.

Let the vertices at the top, starting at the top left hand corner, be a, b and c . Then the claim is that no multiple of

$$\alpha = \langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle$$

is a boundary (we have already seen that every 1-cycle is a multiple of α).

Suppose we write down a 2-chain

$$\beta = \sum a_i \sigma_i.$$

We suppose that

$$m\alpha = \partial_2 \beta,$$

for some integer m. Consider an edge e at the bottom. This has zero coefficient in α. Since e is at the bottom there is only one 2-simplex σ that contains e. But then the coefficient of σ in β must be zero. Thus three 2-simplices must have zero coefficient in β, let's say σ_1, σ_2 and σ_3, corresponding to the three edges on the bottom.

Now consider the three simplices one level up. Pick a diagonal edge f in the bottom row. This has coefficient zero in α. Let τ be the other 2-simplex that contains f (that is, not one of σ_1, σ_2 and σ_3). Then the coefficient of τ in β must be zero. In this way we can show that the coefficient in β of three more simplices, σ_4, σ_5 and σ_6, must be zero.

Now consider the middle horizontal line. This consists of three edges, which belong to σ_4, σ_5 and σ_6. Since these edges don't appear in α, we can show three more simplices σ_7, σ_8 and σ_9 must have coefficient zero in β.

Finally, there are three remaining diagonal edges, that have zero coefficient in α. This forces the coefficients of the last three 2-simplices to be zero. Thus $m = 0$.

It follows that α is a free generator (no multiple is zero) of the first homology. Thus

$$H_1(\mathcal{K}) \simeq \mathbb{Z}.$$

There is a more indirect argument using the rank. The chain complex starts off as

$$0 \longrightarrow \mathbb{Z}^{12} \longrightarrow \mathbb{Z}^{22} \longrightarrow \mathbb{Z}^{9} \longrightarrow 0.$$

We already saw there are no 2-cycles. It follows that

$$B_1(\mathcal{K}) = \mathbb{Z}^{12},$$

since the whole of $\mathbb{Z}^{12}$ injects into its image. We know that

$$H_0(\mathcal{K}) = \mathbb{Z}.$$

Since

$$C_0(\mathcal{K}) = \mathbb{Z}^{9},$$

it follows that

$$B_0(\mathcal{K}) \simeq \mathbb{Z}^{8}.$$

Thus

$$C_1(\mathcal{K}) = \mathbb{Z}^{13}$$

(there is a version of rank-nullity for maps of finitely generated abelian groups. It is hard to keep track of torsion, but the rank behaves as it does for a vector space).

It follows that the quotient

$$H_1(\mathcal{K})$$

has rank at least one.

Since α is the only generator, it follows that rank is one and α is the generator.