4. Homology

In what follows we assume that all simplicial complexes are ordered.

Definition 4.1. *Let $\mathcal{K}$ be a simplicial complex.*

*An n-**chain** is a formal sum of simplexes with integer coefficients:*

$$\sum_{\sigma\in\mathcal{K}} n_\sigma \sigma,$$

where $n_\sigma \in \mathbb{Z}$.

$C_n(\mathcal{K})$ is the set of all n-chains. It is a group with the obvious addition.

More formally, $C_n(\mathcal{K})$ is the free group generated by the n-simplices. If $\mathcal{K}$ is the simplicial complex associated to a triangle σ, with sides σ_0, σ_1 and σ_2, and vertices v_0, v_1 and v_2, then

$$7\sigma \qquad 3\sigma_0 - 5\sigma_1 + 2\sigma_2 \qquad \text{and} \qquad -v_0 - 3v_1 + 10v_2$$

are examples of 2-chains, 1-chains and 0-chains.

We now want a way to get from n-chains to $(n-1)$-chains,

$$\partial_n\colon C_n(\mathcal{K}) \longrightarrow C_{n-1}(\mathcal{K}).$$

It is enough to define $\partial_n\sigma$, where σ is an n-simplex, and then extend linearly. One way to get an $(n-1)$-chain is to simply to drop a vertex. Instead of picking which vertex to drop, we simply drop all of them.

The most obvious thing to do is simply take the sum over the $(n+1)$ ways to drop a vertex. There are two big hints that this is not quite the right thing to do. The first is Euler's formula, which works because there is an alternating sum. The second comes from forms. One key fact about forms is that $d^2 = 0$. By analogy we want

$$\partial^2 = \partial_{n-1} \circ \partial_n = 0.$$

But if we want this we must have cancelling and if we want cancelling we need some negative signs.

We use the orientation to decide the sign. Compatible means plus and incompatible means minus. A little notation:

$$p_0, p_1, \dots, \hat{p}_i, \dots, p_n$$

means we omit the ith vertex. The corresponding simplex comes with the coefficient $(-1)^i$.

Example 4.2. *Let us compute ∂_n for an n-simplex, where $n \leq 3$.*

If we have a 1-simplex

$$\sigma = \langle v_0, v_1\rangle \qquad \text{then} \qquad \partial_1\sigma = v_1 - v_0.$$

If we have a 2-simplex

$$\sigma = \langle v_0, v_1, v_2\rangle \qquad \text{then} \qquad \partial_2\sigma = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle.$$

If we have a 3-simplex

$$\sigma = \langle v_0, v_1, v_2, v_3\rangle \qquad \text{then} \qquad \partial_3\sigma = \langle v_1, v_2, v_3\rangle - \langle v_0, v_2, v_3\rangle + \langle v_0, v_1, v_3\rangle - \langle v_0, v_1, v_2\rangle.$$

Definition-Lemma 4.3. *For every n there is a group homomorphism*

$$\partial_n \colon C_n(\mathcal{K}) \longrightarrow C_{n-1}(\mathcal{K}).$$

We define

$$\partial_n\langle p_0, p_1, \dots, p_n\rangle = \sum_i (-1)^i \langle p_0, p_1, \dots, \hat{p}_i, \dots, p_n\rangle$$

and extend linearly.

The composition

$$\partial^2 = \partial_{n-1} \circ \partial_n \colon C_n(\mathcal{K}) \longrightarrow C_{n-2}(\mathcal{K})$$

is the zero map.

Proof. Everything is clear, apart from the last statement.

Since the elements of $C_n(\mathcal{K})$ are combinations of n-simplices, it suffices to check that

$$\partial^2\sigma = 0,$$

for any n-simplex. Now $\partial^2\sigma$ is a sum over all faces of σ of codimension two. We just need to check that the coefficient of any such face is zero.

If

$$\sigma = \langle p_0, p_1, \dots, p_n\rangle$$

then a face τ of dimension $(n-2)$ is the same as dropping two vertices, let's say p_i and p_j. There is no harm in assuming $i < j$. When we expand

$$\partial^2\sigma = \partial^2\langle p_0, p_1, \dots, p_n\rangle.$$

there are two ways to drop p_i and p_j. First we could drop p_i and then p_j, or we could first drop p_j and then p_i.

If we drop p_i first and we expand $\partial_n\sigma$ then we get

$$\langle p_0, p_1, \dots, \hat{p}_i, \dots, p_n\rangle \qquad \text{with coefficient} \qquad (-1)^i.$$

Now consider what happens if we drop p_j. Note that the position of p_j is now the $(j-1)$th position, since we dropped p_i and p_i comes before p_j. Thus the coefficient of τ, when we compute

$$\partial_{n-1}\langle p_0, p_1, \dots, \hat{p}_i, \dots, p_n\rangle$$

is $(-1)^{j-1}$. Putting all of this together we get coefficient

$$(-1)^{i+j-1}$$

in front of τ, when we first drop p_i and then p_j.

Now suppose we first drop p_j first. If we expand $\partial_n \sigma$ then we get

$$\langle p_0, p_1, \dots, \hat{p_j}, \dots, p_n \rangle \qquad \text{with coefficient} \qquad (-1)^j.$$

Now consider what happens if we drop p_i. Note that the position of p_i did not change since p_j comes after p_i. Thus the coefficient of τ, when we compute

$$\partial_{n-1}\langle p_0, p_1, \dots, \hat{p_j}, \dots, p_n \rangle$$

is $(-1)^i$. Putting all of this together we get coefficient

$$(-1)^{i+j}$$

in front of τ, when we first drop p_j and then p_i.

But then the coefficient of τ in $\partial^2 \sigma$ is

$$\begin{aligned} (-1)^{i+j-1} + (-1)^{i+j} &= -1(-1)^{i+j} + (-1)^{i+j} \\ &= 0. \end{aligned}$$

□

Example 4.4. *Let suppose we have a 3-simplex $\sigma = \langle v_0, v_1, v_2, v_3 \rangle$.*

Let's check that $\partial^2 \sigma = 0$. We have

$$\begin{aligned} \partial^2 \sigma &= \partial_2 \partial_3 \sigma \\ &= \partial_2(\langle v_1, v_2, v_3 \rangle - \langle v_0, v_2, v_3 \rangle + \langle v_0, v_1, v_3 \rangle - \langle v_0, v_1, v_2 \rangle) \\ &= \partial_2 \langle v_1, v_2, v_3 \rangle - \partial_2 \langle v_0, v_2, v_3 \rangle + \partial_2 \langle v_0, v_1, v_3 \rangle - \partial_2 \langle v_0, v_1, v_2 \rangle \end{aligned}$$

We have

$$\begin{aligned} \partial_2 \langle v_1, v_2, v_3 \rangle &= \langle v_2, v_3 \rangle - \langle v_1, v_3 \rangle + \langle v_1, v_2 \rangle \\ -\partial_2 \langle v_0, v_2, v_3 \rangle &= -\langle v_2, v_3 \rangle + \langle v_0, v_3 \rangle - \langle v_0, v_2 \rangle \\ \partial_2 \langle v_0, v_1, v_3 \rangle &= \langle v_1, v_3 \rangle - \langle v_0, v_3 \rangle + \langle v_0, v_1 \rangle \\ -\partial_2 \langle v_0, v_1, v_2 \rangle &= -\langle v_1, v_2 \rangle + \langle v_0, v_2 \rangle - \langle v_0, v_1 \rangle. \end{aligned}$$

One can check the sum is indeed zero.

It is useful to slightly repackage what we have so far. Given a simplicial complex $\mathcal{K}$ we have defined the group $C_n(\mathcal{K})$ of n-chains and we have connecting maps

$$\partial_n \colon C_n(\mathcal{K}) \longrightarrow C_{n-1}(\mathcal{K})$$

and the composition of two such maps is zero. Any such is refered to as a chain complex. It is customary to write out the whole complex and to add the zero map ∂_0 at the end:

$$\cdots \longrightarrow C_n(\mathcal{K}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{K}) \longrightarrow \cdots \longrightarrow C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0.$$

Definition 4.5. *Let $\mathcal{K}$ be a simplicial complex. An n-chain $\alpha \in C_n(\mathcal{K})$ is called an n-**cycle** if $\partial_n \alpha = 0$. An n-chain $\alpha \in C_n(\mathcal{K})$ is called an n-**boundary** if we can find an $(n+1)$-chain $\beta \in C_{n+1}(\mathcal{K})$ such that $\partial_{n+1}\beta = \alpha$.*

In the context of forms, cycles correspond to closed forms and boundaries to exact forms.

The following is very easy but it is also important enough to make a:

Lemma 4.6. *Let $\mathcal{K}$ be a simplicial complex.*

If $\alpha \in C_n(\mathcal{K})$ is a boundary then it is a cycle.

Proof. By assumption we can find $\beta \in C_{n+1}(\mathcal{K})$ such that $\partial_{n+1}\beta = \alpha$.

In this case

$$\begin{aligned}\partial_n \alpha &= \partial_n \circ \partial_{n+1}\beta \\ &= \partial^2 \beta \\ &= 0.\end{aligned}$$ □

Definition-Lemma 4.7. *Let $\mathcal{K}$ be a simplicial complex.*

The set of all n-cycles $Z_n(\mathcal{K})$ is a subgroup of $C_n(\mathcal{K})$ and the set of all boundaries $B_n(\mathcal{K})$ is a subgroup of $Z_n(\mathcal{K})$.

Proof. $Z_n(\mathcal{K})$ is the kernel of the map

$$\partial_n \colon C_n(\mathcal{K}) \longrightarrow C_{n-1}(\mathcal{K})$$

and $B_n(\mathcal{K})$ is the image of the map

$$\partial_{n+1} \colon C_{n+1}(\mathcal{K}) \longrightarrow C_n(\mathcal{K}).$$ □

Note that $Z_n(\mathcal{K})$ is abelian. Thus

$$B_n(\mathcal{K}) \subset Z_n(\mathcal{K})$$

is automatically a normal subgroup. We now come to the key definition of the first half of the course:

Definition 4.8. *Let $\mathcal{K}$ be a simplicial complex.*

*The nth **homology** group of $\mathcal{K}$, denoted $H_n(\mathcal{K})$, is the quotient group*

$$\frac{Z_n(\mathcal{K})}{B_n(\mathcal{K})}.$$

In words, the nth homology group is the group of n-cycles modulo the subgroup of n-boundaries. In particular $H_n(\mathcal{K})$ is non-zero if there are cycles which aren't boundaries.