3. Barycentric coordinates, orderings and orientations

We would first like to show that two simplices are homeomorphic if they have the same dimension. The easiest way to do this is to introduce barycentric coordinates.

First some convenient notation. If $p_0, p_1, \dots, p_m$ are points in linear general position in $\mathbb{R}^n$, where $m \leq n$ then

$$\sigma = \langle p_0, p_1, \dots, p_m \rangle$$

is the simplex with vertices $p_0, p_1, \dots, p_m$. The ordering of the points matters in this representation.

Definition-Lemma 3.1. *Let*

$$\sigma = \langle p_0, p_1, \dots, p_m \rangle$$

be a simplex in $\mathbb{R}^n$.

Then every point $p \in \sigma$ *has a unique representation*

$$p = \sum \lambda_i p_i$$

where

$$\lambda_i \in [0,1] \qquad \textit{and} \qquad \sum \lambda_i = 1.$$

The numbers

$$\lambda_0, \lambda_1, \dots, \lambda_m$$

are called the ***barycentric coordinates*** *of* p.

Proof. We first show the existence of such an expression.

Let

$$C = \{ \sum \lambda_i p_i \mid \lambda_i \in [0,1] \text{ and } \sum \lambda_i = 1 \}.$$

Suppose that p and q belong to C. Then we may find $\lambda_0, \lambda_1, \dots, \lambda_m$ and $\mu_0, \mu_1, \dots, \mu_m$ such that

$$p = \sum \lambda_i p_i \qquad \text{and} \qquad q = \sum \mu_i p_i.$$

Let $t \in [0,1]$. Then

$$\begin{aligned} tp + (1-t)q &= t(\sum \lambda_i p_i) + (1-t) \sum \mu_i p_i \\ &= \sum t\lambda_i p_i + (1-t)\mu_i p_i \\ &= \sum (t\lambda_i + (1-t)\mu_i) p_i \\ &= \sum \nu_i P_i. \end{aligned}$$

Note that

$$\nu_i = t\lambda_i + (1-t)\mu_i \in [0,1]$$

and

$$\begin{aligned}\sum \nu_i &= \sum (t\lambda_i + (1-t)\mu_i) \\ &= t\sum \lambda_i + (1-t)\sum \mu_i \\ &= t + (1-t) \\ &= 1.\end{aligned}$$

Thus $tp + (1-t)q \in C$ and so C is convex. As $p_i \in C$ and σ is the smallest convex setting containing $p_0, p_1, \dots, p_m$ it follows that

$$\sigma \subset C.$$

Suppose that $p \in C$. Let

$$\mu_i = \frac{\lambda_i}{t} \qquad \text{where} \qquad t = \sum_{i<m} \lambda_i = 1 - \lambda_m.$$

Then

$$\mu_i \in [0,1] \qquad \text{and} \qquad \sum_{i<m} \mu_i = 1.$$

Thus

$$q = \sum_i \mu_i p_i \in \sigma$$

by induction on m. On the other hand, as

$$\lambda_m = (1-t)$$

and σ is convex it follows that

$$p = tp + (1-t)p_m \in \sigma.$$

Thus $\sigma = C$.

Now we consider uniqueness. Suppose that we can write

$$\sum \lambda_i p_i = \sum \mu_i p_i.$$

Let $t_i = \lambda_i - \mu_i$. Then

$$\sum t_i = 0 \qquad \text{and} \qquad \sum t_i p_i = 0.$$

Let $v_i = p_i - p_0$. Then

$$\begin{aligned} 0 &= \sum t_i p_i \\ &= \sum t_i p_i - 0 p_0 \\ &= \sum t_i p_i - \sum t_i p_0 \\ &= \sum t_i (p_i - p_0) \\ &= \sum_{i>0} t_i v_i. \end{aligned}$$

But the vectors $v_1, v_2, \dots, v_m$ are linearly independent, as the points $p_0, p_1, \dots, p_m$ are in linear general position. Thus $t_i = 0$ for all $i > 0$. But then $\lambda_i = \mu_i$ and so barycentric coordinates are unique. □

Let

$$e_1, e_2, \dots, e_n$$

be the standard basis of $\mathbb{R}^n$ (so that e_i has a 1 in the ith position and zeroes everywhere else).

Definition 3.2. *Let $e_0, e_1, \dots, e_n$ be the points corresponding to the standard basis of $\mathbb{R}^{n+1}$. The **standard n-simplex**, denoted Δ^n, is the corresponding simplex*

$$\Delta^n = \langle e_0, e_1, \dots, e_n \rangle$$

Note that the standard simplex lives in the affine hyperplane

$$\sum x_i = 1$$

and barycentric coordinates are just the same as the usual coordinates.

Lemma 3.3. *Any two m-simplices are homeomorphic.*

Proof. Let $\sigma \subset \mathbf{R}^p$ and $\tau \subset \mathbf{R}^q$ be two m simplices. Define a map

$$f \colon \sigma \longrightarrow \tau \qquad \text{by the rule} \qquad p = \sum \lambda_i p_i \longrightarrow \sum_i \lambda_i q_i,$$

where $\lambda_0, \lambda_1, \dots, \lambda_m$ are the barycentric coordinates of p. (3.1) implies that f is a well-defined bijection. We have to check that f is continuous. It is easiest, and enough, to assume that $\sigma = \Delta^n$ is the standard simplex.

In this case note that the coordinates of p in $\mathbb{R}^{n+1}$ are precisely the barycentric coordinates of p. In this case the coordinates of f are then linear functions of the coordinates of p, so that f is continuous. It then follows that f is a homeomorphism, σ is compact and τ is Hausdorff. □

Now let us turn to orderings and orientations. First orderings. Note that an ordering of the vertices of σ automatically gives an ordering of the vertices of any face, since we just throw away some vertices but otherwise keep the ordering.

Now suppose we are given a simplicial complex $\mathcal{K}$. If we randomly order the vertices of each maximal simplex then we might run into trouble. One edge might belong to two different triangles and receive a different ordering from each triangle. There is an easy fix here. Pick an ordering of the vertices of $\mathcal{K}$. This automatically orders every simplex in $\mathcal{K}$ and it is clear that this ordering is therefore automatically compatible.

Now let us turn to orientations. Note that any two orderings of a simplex differ by a permutation of the vertices, an element of the symmetric group S_n, where n is the number of vertices. The simplest permutation is a transposition, something that swaps just two vertices. Permutations are divided into two types, even and odd. The evens are a product of an even number of transpositions and the odds are a product of an odd number of transpositions (the key result is then to check that no permutation is both a product of an odd and of an even number of transpositions). The even permutations are a subgroup A_n of S_n, called the alternating group.

The action of A_n on the set of all orderings divides orderings into two orbits. These orbits are called **orientations**. An orientation of a 1-simplex is the same as an ordering. We can think of an orientation as an arrow from one vertex to the other. One geometrically appealing way to think of an orientation of a triangle is the choice of a way to go around the triangle (but then there is no ordering of the edges which induces these orderings).

Similarly an orientation of a tetrahedron gives compatible orientations of its sides, which then give compatible orientations of the edges.