23. The fundamental group of the circle

Theorem 23.1. *The fundamental group of the circle is infinite cyclic.*

Proof. We saw in Lecture 20 that

$$p\colon \mathbb{R} \longrightarrow S^1 \qquad \text{given by} \qquad p(t) = e^{2\pi i t}$$

realises $\mathbb{R}$ as a covering space of S^1. $\mathbb{R}$ is simply connected (for example, it is contractible).

The fibre over 1 is the set of integers. Thus the fundamental group is certainly countable. Pick two points $z_0 = a$ and $z_1 = b$ in the fibre over $x_0 = 1$. Then a and b are integers. Consider the path

$$u\colon I \longrightarrow \mathbb{R} \qquad \text{given by} \qquad u(t) = tb,$$

starting at $y_0 = 0$ and ending at b. Then

$$\gamma = p \circ u\colon I \longrightarrow S^1 \qquad \text{is given by} \qquad t \longrightarrow e^{2\pi i t b}.$$

The lift of γ starting at a is

$$\tilde{\gamma}\colon I \longrightarrow \mathbb{R} \qquad \text{given by} \qquad t \longrightarrow a + tb.$$

This ends at $a + b$. Thus we get the integers under addition. □

Theorem 23.2. *The disc $D^2 \subset \mathbb{R}^2$ does not retract to $S^1 \subset D^2$.*

Proof. Suppose not, suppose there is a continuous function

$$r\colon D^2 \longrightarrow S^1$$

such that $r \circ i = \mathrm{id}_{S^1}$, where $i\colon S^1 \longrightarrow D^2$ is the natural inclusion.

Then there are three maps

$$i_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(D, 1) \quad r_*\colon \pi_1(D, 1) \longrightarrow \pi_1(S^1, 1) \quad (\mathrm{id}_{S^1})_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1).$$

Note that $\pi_1(D, 1) = 0$, as D is contractible. Thus the first map is zero. Similarly the second map is zero. Consider the composition

$$\begin{aligned} 0 &= r_* \circ i_* \\ &= (r \circ i)_* \\ &= (\mathrm{id}_{S^1})_* \\ &= \mathrm{id}_*. \end{aligned}$$

This is not possible, as the fundamental group of the circle is non-trivial. □

Corollary 23.3 (Brouwer fixed point theorem)**.** *Every continuous function $f\colon D \longrightarrow D$ has a fixed point.*

Proof. Suppose not. Note that the function

$$l\colon D \times D \setminus \Delta \longrightarrow D$$

which sends (x, y) to the point where the line from y to x ($\neq y$) meets S^1, is continuous.

Let r be the composition of

$$D \longrightarrow D \times D \setminus \Delta \qquad \text{given by} \qquad x \longrightarrow (x, f(x))$$

and l. Then r is continuous and $r \circ i$ is the identity. Thus r is a retraction, which is impossible, by (23.2).

Therefore f has a fixed point. □

Theorem 23.4 (Fundamental Theorem of Algebra)**.** *Every polynomial of degree n with complex coefficients has n roots, counted with multiplicity.*

Proof. Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$$

be a monic polynomial with no roots. We have to show $n = 0$. Let

$$r = \max(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|, 1).$$

On the circle $|z| = r$ of radius r we have the following estimates

$$\begin{aligned} |z^n| &= |z|^{n-1} \cdot r \\ &\geq |z|^{n-1}(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|) \\ &\geq |a_{n-1}z^{n-1} + \cdots + a_0|. \end{aligned}$$

Thus the polynomial

$$p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$$

has no roots on the circle $|z| = r$ for any $t \in [0, 1]$.

Consider the function

$$F\colon I^2 \longrightarrow S^1 \qquad \text{given by} \qquad F(s, t) = \frac{w}{|w|}$$

where

$$w = \frac{p_t(r \cdot e^{2\pi i s})}{p_t(r)}.$$

F is continuous, as w is never zero. Thus F is a homotopy from the loop γ

$$F(s, 0) = e^{2\pi i n s} \qquad \text{to the loop} \qquad F(s, 1) = f_r(s) = \frac{w}{|w|}$$

where

$$w = \frac{p(r \cdot e^{2\pi i s})}{p(r)}.$$

Thus

$$\begin{aligned}[f_r] &= [\gamma] \\ &= n.\end{aligned}$$

On the other hand, f_r is homotopic to f_0, the constant function 1, as we are assuming that $p(z)$ has no roots. Thus $n = 0$. □