22. The universal cover

Given $f\colon X \longrightarrow Y$ continuous the **fibre** over y is the set $f^{-1}(y)$.

Definition 22.1. *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space. Given a path $\gamma\colon x_0 \rightsquigarrow x_1$ and a point y_0 in the fibre over x_0 let*

$$y_1 = y_0 \bullet \gamma$$

be the endpoint of a lift of γ starting at y_0.

1. *$y_0 \bullet \gamma$ only depends on the homotopy class of γ.*
2. *$\bullet$ respects concatenation.*

In particular there is a right action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$,

$$p^{-1}(x_0) \times \pi_1(X, x_0) \longrightarrow p^{-1}(x_0) \qquad \text{given by} \qquad y_0 \bullet [\gamma] = [y_0 \bullet \gamma].$$

Proof. (1) is proved in (21.5).

Suppose that $\gamma_i\colon x_i \rightsquigarrow x_{i+1}$ are two paths, $i = 0$, 1.

Pick y_0 in the fibre over x_0. Let

$$y_1 = y_0 \bullet \gamma_0 \qquad \text{and let} \qquad y_2 = y_1 \bullet \gamma_1.$$

Then y_1 is the endpoint of the lift $\tilde{\gamma}_0$ of γ_0 starting at y_0 and y_2 is the endpoint of the lift $\tilde{\gamma}_1$ of γ_1 starting at y_1.

Note that (21.2) implies $\tilde{\gamma}_0 \cdot \tilde{\gamma}_1$ is the lift of $\gamma_0 \cdot \gamma_1$ starting at y_0. Hence

$$y_2 = y_0 \bullet (\gamma_0 \cdot \gamma_1). \qquad \square$$

Theorem 22.2. *Let $p\colon \tilde{X} \longrightarrow X$ be a covering space of a path connected space X and let $x_0 \in X$ be a point.*

1. *The action of $\pi_1(X, x_0)$ on the fibre over x_0 is transitive if and only if $\tilde{X}$ is path connected.*
2. *The stabiliser of a point y_0 of the fibre over x_0 is the image of*
$$p_*\colon \pi_1(\tilde{X}, y_0) \longrightarrow \pi_1(Y, x_0).$$
3. *If $\tilde{X}$ is path connected then the fibre over x_0 is in bijection with the right cosets of $p_*(\pi_1(\tilde{X}, y_0))$ in $\pi_1(X, x_0)$.*

Proof. We first prove (1). Suppose that $\tilde{X}$ is path connected. Let y_0 and y_1 be two points in the fibre. Pick a path u from y_0 to y_1 and let γ be the image of u. Then γ is a loop based at x_0 and the lift $\tilde{\gamma}$ of γ starting at y_0 must be u. But then

$$y_1 = y_0 \bullet [\gamma]$$

and the action is transitive. Now suppose that the action is transitive. Let y_0 and y_1 be two points in a fibre. Pick a loop γ such that

$$y_1 = y_0 \bullet [\gamma].$$

Then the lift $\tilde{\gamma}$ of γ starting at y_0 is a path from y_0 to y_1. Thus the whole fibre of p over x_0 lies in the same path component. Now suppose that y_0 and y_1 are two points of $\tilde{X}$ and y_0 belongs to the fibre over x_0. Let x_1 be the image of y_1. Pick a path u from x_0 to x_1. Lift u to a path v ending at y_1. Then v starts at some point y_2 of the fibre. But then y_0 and y_1 must belong to the same path component. This is (1).

Suppose that $\gamma \in \pi_1(X, x_0)$ and

$$y_0 = y_0 \bullet [\gamma].$$

Let $\tilde{\gamma}$ be a lift of γ starting at y_0. Then $\tilde{\gamma}$ ends at y_0 a and so $\tilde{\gamma}$ is a loop based at y_0. As $\gamma = p \circ \tilde{\gamma}$, we have

$$\begin{aligned} [\gamma] &= [p \circ \tilde{\gamma}] \\ &= p_*[\tilde{\gamma}] \in p_*(\pi_1(\tilde{X}, y_0)). \end{aligned}$$

Now suppose that $\delta \in \pi_1(\tilde{X}, y_0))$ and let $\gamma = p \circ \delta$. Then γ is a loop based at x_0 and δ is the lift starting at y_0. Thus

$$y_0 = y_0 \bullet [\gamma]$$

so that $p_*\delta$ stabilises y_0. This is (2).

(3) holds for any transitive action. □

Definition 22.3. *We say that a covering map $p\colon \tilde{X} \longrightarrow X$ is the* ***universal cover*** *if $\tilde{X}$ is simply connected.*

Corollary 22.4. *Let $p\colon \tilde{X} \longrightarrow X$ be the universal cover.*

A point y_0 in the fibre determines a bijection between

$$l\colon \pi_1(X, x_0) \longrightarrow p^{-1}(x_0) \qquad \text{given by} \qquad [\gamma] \longrightarrow y_0 \bullet [\gamma].$$

Proof. As $\tilde{X}$ is path connected, (1) of (22.2) implies that l is surjective. As $\tilde{X}$ has trivial fundamental group, (2) implies that l is injective. □

We can use (22.4) to figure out the fundamental group. This is a little bit subtle, as the RHS of the bijection is a set, not a group. Let z_0 and z_1 be two points in the fibre. The induced group structure on the fibre determined by l is given by

$$y_1 = l(l^{-1}(z_0) \cdot l^{-1}(z_1)).$$

How do we compute this in practice?

- Pick a path $u\colon y_0 \rightsquigarrow z_1$.
- Let $\gamma = p \circ u$ be the corresponding loop based at x_0.
- $y_1 = z_0 \bullet [\gamma]$.