20. Covering spaces

Definition 20.1. *A* ***covering space*** *of a topological space X is a pair $(\tilde{X}, p)$ consisting of a topological space $\tilde{X}$ and a continuous map $p\colon \tilde{X} \longrightarrow X$ such that every point of $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically down to U. p is called a* ***covering map****.*

Note that we can write

$$p^{-1}(U) = \coprod_{\alpha \in \Lambda} V_\alpha$$

where each V_α is an open subset of $\tilde{X}$ and

$$p|_{V_\alpha}\colon V_\alpha \longrightarrow U$$

is a homeomorphism. We will call any such U **evenly covered**,

Example 20.2. *A homeomorphism is a covering map.*

Every open set is evenly covered.

Example 20.3. *If $p\colon \tilde{X} \longrightarrow X$ is a cover of X and $q\colon \tilde{Y} \longrightarrow Y$ is a cover of Y then*

$$p \times q\colon \tilde{X} \times \tilde{Y} \longrightarrow X \times Y$$

is a cover of the product.

Example 20.4. *Let*

$$S^1 = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}.$$

Consider the map

$$p\colon \mathbb{R} \longrightarrow S^1 \qquad \text{given by} \qquad p(t) = e^{2\pi i t}.$$

p is certainly continuous. Let

$$U_{y>0} = \{\, x + iy \in S^1 \mid y > 0 \,\}.$$

Then

$$p^{-1}(U_{y>0}) = \coprod_{j \in \mathbb{Z}} (j, j + 1/2).$$

Now

$$p|_{(j,j+1/2)}\colon (j, j + 1/2) \longrightarrow U_{y>0}$$

is continuous and it is a bijection. In fact it is a homeomorphism. There are two ways to see this. The first is simply to observe that this map extends to the closed interval $[j, j + 1/2]$, which is compact and this extension is still a bijection. As the image is Hausdorff it follows that the extension is a homeomorphism. But then the original map is a homeomorphism.

Or we could simply write down the inverse map

$$U_{y>0} \longrightarrow (j, j+1/2) \qquad \text{given by} \qquad x+iy \longrightarrow j + \frac{\cos^{-1}(x)}{2\pi}.$$

Note that we can repeat a similar argument with the three open sets $U_{y<0}$, $U_{x>0}$ and $U_{x<0}$. As these cover S^1, it follows that p is a covering map.

There is another way to view all of this which is geometrically quite appealing. We can embed $\mathbb{R}$ into $\mathbb{R}$ as a helix,

$$\mathbb{R} \longrightarrow \mathbb{R}^3 \qquad \text{given by} \qquad t \longrightarrow (\cos 2\pi t, \sin 2\pi t, t).$$

In fact this is really just the graph of the map above (although we switched the order of domain and range. The z-variable corresponds to the t-variable). Projection down to the xy-plane is simply is a map onto the circle. Projection down (across) to the z-axis is a homeomorphism. The inverse map composed with projection down to the xy-plane is the map p above.

If we take an open set in S^1, for example the set $U_{y>0}$, it is easy to see that the inverse image in the helix is a disjoint union of open sets, all homeomorphic to $U_{y>0}$.

Example 20.5. *There is a cover from $\mathbb{R}^2$ to the torus.*

Just use the previous two examples and the fact that $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$ and the torus is $S^1 \times S^1$.

Let us suppose we take two circles in the torus. Fix a point (p_0, q_0) and consider the union

$$S^1 \times \{ q_0 \} \cup \{ p_0 \} \times S^1.$$

These circles both pass through (p_0, q_0) and there is no other point in common.

It is interesting to consider the inverse image of this circle. Notice that in the plane $\mathbb{R}$, each grid unit square

$$[i, i+1] \times [j, j+1]$$

gets mapped onto the torus. If we choose $p_0 = q_0 = 1$, then the inverse image of (p_0, q_0) are the grid points, the points with integer coordinates

$$\{ (i, j) \mid i, j \in \mathbb{N} \}.$$

The circles one way are given by the horizontal lines $\mathbb{R} \times \{ j \}$ and the circles the other way are given by the vertical lines $\{ j \} \times \mathbb{R}$.

It is not hard to see that the union of all these horizontal and vertical lines gives a covering space of the union of the two circles.

Example 20.6. *Consider the natural map*

$$p\colon S^2 \longrightarrow \mathbb{RP}^2$$

which sends a point of S^2 to the line it spans. If take a point x of the real projective plane, then there are two points v and $-v$ both of which map to x. In fact p is a quotient map.

p is a two to one map. Let $V_{z>0}$ be all points of S^2 whose z-coordinate is positive. Similarly let $V_{z<0}$ be the set of all points whose z-coordinate is negative. Let U be the image of $V_{z>0}$. The inverse image of U is the union $V_{z>0} \cup V_{z<0}$. As this is open, U is open. The restriction of p to both $V_{z>0}$ and $V_{z<0}$ is a bijection and it easy to argue the restriction is a homeomorphism. Thus U is evenly covered.

Replacing z by x and by y it follows that p is a covering map. We call p a double cover.