2. Simplicial complexes

Definition 2.1. *$\Lambda \subset \mathbb{R}^n$ is an* ***affine linear subspace*** *if it is the translate of a linear subspace V of the vector space $\mathbb{R}^n$ (or it is empty).*

The dimension of Λ is the same as the dimension of the linear subspace,

$$\dim \Lambda = \dim V,$$

(or it is minus infinity).

Note that an affine linear space is the same as the set of solution of a system of linear equations

$$\Lambda = \{\, x \in \mathbb{R}^n \mid Ax = b \,\}$$

where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. The dimension of Λ is the number of free variables.

For example,

$$\{\, (x, y) \in \mathbb{R}^2 \mid y = 5x - 6 \,\}$$

is a line in the plane.

Lemma 2.2. *The intersection of affine linear subspaces is an affine linear subspace.*

Proof. Let

$$\{\, \Lambda_i \mid i \in J \,\}$$

be a collection of affine linear subspaces and let

$$\Lambda = \bigcap_{i \in J} \Lambda_i$$

be their intersection. If Λ is empty there is nothing to prove. Otherise pick a common point p and consider translating by $-p$. We get linear subspaces V_i and Λ is a translate of the intersection

$$V = \bigcap_{i \in J} V_i. \qquad \square$$

In terms of equations we just impose all of the equations defining Λ_i.

Definition-Lemma 2.3. *Let X be a subset of $\mathbb{R}^n$.*

The ***span*** *of X is the smallest affine linear space Λ containing X.*

Proof. Let

$$\{\, \Lambda_i \mid X \subset \Lambda_i \,\}$$

be the collection of all affine linear spaces containing X. The span is the intersection. $\square$

Definition 2.4. *A collection of k points $p_1, p_2, \ldots, p_k$ in $\mathbb{R}^n$ is in linear general position if given any subcollection $q_0, q_1, \ldots, q_m$ of $m+1$ points, where $m \leq n$, the span of these points is m-dimensional.*

For example p and q are in linear general position if they don't coincide (or $n = 0$) p, q and r are in linear general position if they are not collinear (or $n = 1$ and no two points coincide or $n = 0$).

On the other hand a million points in $\mathbb{R}^3$ are in linear general position if no four are coplanar (obviously no set of points will ever span a space of dimension four or more and if four points don't lie in a plane then in fact no three points are collinear and no two points coincide).

Definition-Lemma 2.5. *A subset $X \subset \mathbb{R}^n$ is* ***convex*** *if for any two points p and $q \in X$ the whole line segment between p and q lies in X, that is if $\lambda \in [0, 1]$ then*

$$\lambda p + (1 - \lambda)q \in X.$$

The ***convex hull*** *of a set X is the smallest convex C set that contains X.*

Definition 2.6. *Let $p_0, p_1, \ldots, p_m$ be $(m + 1)$ points in linear general position in $\mathbb{R}^n$, where $m \leq n$.*

*The convex hull σ of these points is an m-**simplex**. The points $p_0, p_1, \ldots, p_m$ are called the* ***vertices*** *of σ.*

A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron.

Definition 2.7. *Let σ be a simplex.*

A ***face*** *τ of σ is the simplex given by a subset of the vertices of σ.*

If σ is a simplex then σ is a face of σ. The other faces are called **proper faces**. Given any simplex σ the vertices of σ are faces of σ.

The other faces of the triangle are its three sides and the other faces of a tetrahedron are it four sides and its six edges.

We now come to the key definition.

Definition 2.8. *A simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ is a finite collection of simplices in $\mathbb{R}^n$ such that*

1. *If $\sigma \in \mathcal{K}$ and τ is a face of σ then $\tau \in \mathcal{K}$.*
2. *If σ and $\tau \in \mathcal{K}$ then either $\sigma \cap \tau$ is empty or the intersection $\rho = \sigma \cap \tau$ is a face of both σ and τ.*

The ***dimension*** *of $\mathcal{K}$ is the largest dimension of a simplex $\sigma \in \mathcal{K}$.*

It is sometimes convenient to emphasise the difference between $\mathcal{K}$, which is just a finite collection of sets, and the **support** of $\mathcal{K}$, denoted

$|\mathcal{K}| \subset \mathbb{R}^n$, which is the union of all of the elements of every simplex in $\mathcal{K}$. Observe that $|\mathcal{K}|$ is a topological space, since it inherits the subspace topology from $\mathbb{R}^n$.

Note that a simplex is a closed and bounded subset of $\mathbb{R}^n$. Thus a simplex is compact. As $|\mathcal{K}|$ is a finite union of simplices it follows that $|\mathcal{K}|$ is always compact. On the other hand $\mathbb{R}^n$ is Hausdorff and so $|\mathcal{K}|$ is Hausdorff.

Definition 2.9. *A **triangulation** of a topological space X is a pair of a simplicial complex $\mathcal{K}$ and a homeomorphism*

$$f\colon |\mathcal{K}| \longrightarrow X.$$

Definition 2.10. *An **annulus** is any topological space*

$$\overline{B}_r(a) \setminus B_s(a) = \{\, (x,y) \in \mathbb{R}^2 \mid s \leq (x-a_1)^2 + (y-a_2)^2 \leq r \,\}$$

where $0 < s < r$ are two real numbers and $a = (a_1, a_2)$.

Example 2.11. *Any annulus has a triangulation.*

Note first that all annuli are homeomorphic. Indeed, by translation we may assume that $a = 0$. Rescaling we may assume that $s = 1$. It is then not hard to arrange for $r = 2$.

In fact all annuli are homeomorphic to the cylinder $S^1 \times [0,1]$ (in fact some topologists just define an annulus to be the cylinder). Just embed $S^1 \times [0,1]$ into $\mathbb{R}^3$ in the obvious way and project from $(0,0,1)$ onto the xy-plane. The image is an annulus and the map is a homeomorphism by the usual arguments.

We will use the notation $I = [0,1]$. So it is enough to triangulate $S^2 \times I$. Recall that this is a quotient of the unit square I^2, where the two sides given by $x = 0$ and $x = 1$ are identified.

It is easy to triangulate the unit square. Just take the two triangles, one with vertices $(0,0)$, $(0,1)$ and $(1,1)$ and the other with vertices $(1,1)$, $(1,0)$ and $(0,0)$.

This does not quite give a triangulation of $S^1 \times I$, since the two triangles get messed up when we identify. The problem is that $(0,0)$ and $(1,0)$ is a side of one triangle and these two vertices get identified.

There is an easy fix. We just need to make our triangles smaller so that the vertices don't get identified. Cut the unit square into four smaller squares, by inserting the two line segments $(0,1/2)$ to $(1,1/2)$ and $(1/2,0)$ to $(1/2,1)$. Now triangulate each square by inserting a diagonal.

Now when we take the quotient map

$$q\colon I^2 \longrightarrow S^1 \times I$$

the restriction of q to any triangle is a bijection and so it is a homeomorphism.

However we don't quite get a triangulation of the cylinder. Consider the bottom three vertices $(0, 0)$, $(1/2, 0)$ and $(1, 0)$. The two edges $(0, 0)$ to $(1/2, 0)$ and $(1/2, 0)$ to $(1, 0)$ are mapped to distinct edges in the quotient. But $(0, 0)$ and $(1, 0)$ get mapped to the same point p and $(1/2, 0)$ gets mapped to another point q. If this was represented by a simplicial complex then the two edges get mapped to the same line segment from p to q.

There is an easy fix for this problem. Just use more triangles again. In fact we can remove the middle dividing line and replace the middle vertical line by two vertical lines, say $(1/3, 0)$ to $(1/3, 1)$ and $(2/3, 0)$ to $(2/3, 1)$. Now we have three rectangles and we can divide each into two triangles, by adding one of the diagonals. Now imagine I^2 was made of cardboard. We could fold the cardboard along the two vertical lines and close it up along the edges we identify. The induced map is a quotient map. On the other hand the resulting object is locally flat and it is easy made up of simplices (it is basically like a toblerone bar) and so we do have a triangulation.

It is fun to count the number of vertices, edges and faces, v, e and f. The triangulation of I^2 has eight vertices, thirteen edges and six faces.

Now consider how things change when we take the quotient. We pair off two pairs of vertices and so $v = 6$. We pair off one pair of edges and so $e = 12$. Finally no triangles are identified and so $f = 6$.

The number $\chi = v - e + f$ is called the **Euler characteristic**. In our case

$$\begin{aligned}\chi &= v - e + f\\ &= 6 - 12 + 6\\ &= 0.\end{aligned}$$

In fact we can still compute the Euler characteristic for the non-triangulation. The triangulation of I^2 has nine vertices, sixteen edges and eight faces.

Now consider how things change when we take the quotient. We pair off three pairs of vertices and so $v = 6$. We pair off two pair of edges and so $e = 14$. Finally no triangles are identified and so $f = 8$.

If we compute the Euler characteristic then we get

$$\begin{aligned}\chi &= v - e + f\\ &= 6 - 14 + 8\\ &= 0.\end{aligned}$$

We will see later why we get the same answer, even though we don't really even have a triangulation in this case.