19. Simply connected

Theorem 19.1. *Let $f\colon X \longrightarrow Y$ be a homotopy equivalence of two topological spaces X and Y. Let $x_0 \in X$ be a base point.*

Then

$$f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

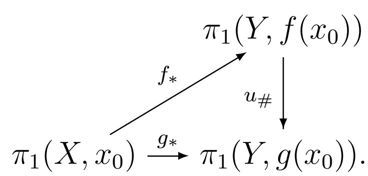
is an isomorphism (of groups).

We will need a:

Lemma 19.2. *Let $F\colon X \times I \longrightarrow Y$ be a homotopy between two continuous maps f and g. Let $x_0 \in X$ be a base point. Let u be the path*

$$u\colon I \longrightarrow Y \qquad \text{given by} \qquad u(t) = F(x_0, t).$$

Then the following diagram commutes



Proof. We have to show that

$$u_\# \circ f_* = g_*.$$

Let γ be a loop based at x_0. Consider the function

$$G\colon I^2 \longrightarrow Y$$

given as the composition of

$$\gamma \times \mathrm{id}_I\colon I^2 \longrightarrow X \times I \qquad \text{and} \qquad F\colon X \times I \longrightarrow Y.$$

We consider two paths in I^2 from $(0, 1)$ to $(1, 1)$. The first is

$$l^+\colon I \longrightarrow I^2 \qquad \text{given by} \qquad s \longrightarrow (s, 1).$$

Or one can the long way around l^-, the concatenation of three paths

$$s \longrightarrow (0, 1 - s) \qquad s \longrightarrow (s, 0) \qquad \text{and} \qquad s \longrightarrow (1, s).$$

We define a homotopy from l^+ to l^-, as paths from $(0, 1)$ to $(1, 1)$ as follows

$$L\colon I^2 \longrightarrow I^2 \qquad \text{given by} \qquad L(s, t) = tl^-(s) + (1 - t)l^+(s).$$

Note that this uses the fact that $I^2 \subset \mathbb{R}^2$ is convex.

It follows that

$$G \circ l^+ \sim G \circ l^-$$

as paths from

$$G(0, 1) = F(x_0, 1) = g(x_0) \qquad \text{to} \qquad G(1, 1) = F(x_0, 1) = g(x_0).$$

On the other hand

$$\begin{aligned}[G \circ l^+] &= [g \circ \gamma] \\ &= g_*([\gamma]),\end{aligned}$$

and

$$\begin{aligned}[G \circ l^-] &= [u^{-1} \cdot (f \circ \gamma) \cdot u] \\ &= u_\#([f \circ \gamma]) \\ &= (u_\# \circ f_*)([\gamma]).\end{aligned}$$ □

Proof of (19.1). By assumption there is a continuous function $g\colon Y \longrightarrow X$ that is a homotopy inverse to f. This means there are homotopies F from $f \circ g$ to id_Y and G from $g \circ f$ to id_X.

Let v be the path

$$v\colon I \longrightarrow X \qquad \text{given by} \qquad t \longrightarrow G(x_0, 1-t).$$

Then v goes from x_0 to $(g \circ f)(x_0)$. Now

$$v_\#\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, (g \circ f)(x_0)).$$

is an isomorphism. On the other hand (19.2) implies that

$$v_\# = (g \circ f)_* = g_* \circ f_*.$$

But then f_* is injective and

$$g_*\colon \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, (g \circ f)(x_0))$$

is surjective. If we can show that g_* is injective as well, then it is an isomorphism and it would follow that $f_* = (g_*)^{-1} \circ v_\#$ is an isomorphism.

To this end, consider the path u

$$u\colon I \longrightarrow Y \qquad \text{given by} \qquad t \longrightarrow F(f(x_0), 1-t).$$

Then u goes from $f(x_0)$ to $(g \circ f)(x_0)$. As before

$$u_\# = (f \circ g)_* = f_* \circ g_*\colon \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, (g \circ f)(x_0)) \longrightarrow \pi_1(Y, ((f \circ g \circ f)(x_0)).$$

It follows that g_* is injective as $u_\#$ is an isomorphism. □

Warning: One can also conclude that f_* is surjective. Beware, however, that the second f_* is not the same as the first f_*! The problem is that the base points changed.

Definition 19.3. *We say that a topological space is* ***simply connected*** *if if is path connected and the fundamental group, based at a point, is trivial.*

Note that if the fundamental group of a path connected space is trivial at one point then it is trivial at all points.

Example 19.4. *A contractible space is simply connected.*

Indeed a contractible space is surely path connected. (19.1) implies that the fundamental group of a contractible space is the same as the fundamental group of a point. But the fundamental group of a point is surely trivial.