18. The fundamental group

Definition 18.1. *A* ***based space*** *is a pair (X, x_0), where X is a topological space and $x_0 \in X$ is a point of X, called the* ***base point****.*

A function f between two based spaces (X, x_0) and (Y, y_0),

$$f\colon (X, x_0) \longrightarrow (Y, y_0)$$

is simply a continuous function

$$f\colon X \longrightarrow Y \qquad \text{such that} \qquad f(x_0) = y_0.$$

A ***based homotopy*** *is simply a homotopy relative to x_0,* (17.3).

It is easy to see that there is a category of based topological spaces.

Definition-Theorem 18.2. *There is a functor π_1 from the category of based topological spaces to the category of groups.*

To every based space (X, x_0), $\pi_1(X, x_0)$ is the group of all loops based at x_0, up to based homotopy. $\pi_1(X, x_0)$ is called the ***fundamental group****.*

To every function f between two based spaces (X, x_0) and (Y, y_0), we define the function

$$\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0) \qquad \text{by the rule} \qquad \pi_1(f)[\gamma] = [f \circ \gamma].$$

Proof. We have to check a number of things. We first check that $\pi_1(X, x_0)$ is a group. Given two loops γ_0 and γ_1 we define

$$[\gamma_0] \cdot [\gamma_1] = [\gamma_0 \cdot \gamma_1].$$

We checked in (17.5) that this is well-defined, that is, the product does not depend on the representative we pick from each equivalence class.

Let

$$e = [c_{x_0}]$$

Then we checked the three axioms for a group (associativity, identity, inverse) in (17.6). Thus $\pi_1(X, x_0)$ is a group.

Now we check that $\pi_1(f)$ is a group homomorphism. Homotopies respect composition and so $\pi_1(f)$ is well-defined,

$$\pi_1(f)[\gamma]$$

does not depend on γ, only on the equivalence class $[\gamma]$. Note that

$$f \circ (\gamma_0 \cdot \gamma_1) = (f \circ \gamma_0) \cdot (f \circ \gamma_1)$$

so that

$$\begin{aligned}\pi_1(f)[\gamma_0]\cdot\pi_1(f)[\gamma_1] &= [f\circ\gamma_0]\cdot[f\circ\gamma_1]\\ &= [(f\circ\gamma_0)\cdot(f\circ\gamma_1)]\\ &= [f\circ(\gamma_0\cdot\gamma_1)]\\ &= \pi_1(f)[\gamma_0\cdot\gamma_1].\end{aligned}$$

Thus $\pi_1(f)$ is a group homomorphism.

π_1 obviously sends the identity to the identity. It also easy to see it respects composition. □

We will use the notation f_* for $\pi_1(f)$. The first thing to do is figure out the relationship between the fundamental group of the same space based at different points.

Definition-Proposition 18.3. *Let X be a topological space. A path $u\colon x_0 \rightsquigarrow x_1$ from x_0 to x_1 induces an isomorphism*

$$u_\#\colon \pi_1(X,x_0)\longrightarrow \pi_1(X,x_1) \qquad \text{given by} \qquad u_\#[\gamma]=[u^{-1}\cdot\gamma\cdot u],$$

satisfying the following properties

1. *If $u\sim v$ as paths from x_0 to x_1 then $u_\# = v_\#$.*
2. *$(c_{x_0})_\# = id_{\pi_1(X,x_0)}$.*
3. *If $v\colon x_1\rightsquigarrow x_2$ then $(u\cdot v)_\# = v_\#\circ u_\#$.*
4. *If $f\colon X\longrightarrow Y$ is a continuous function that sends x_0 to y_0 and x_1 to y_1 then the following diagram commutes*

$$\begin{array}{ccc}\pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,y_0)\\ {\scriptstyle u_\#}\big\downarrow & & \big\downarrow{\scriptstyle (f\circ u)_\#}\\ \pi_1(X,x_1) & \xrightarrow{f_*} & \pi_1(Y,y_1).\end{array}$$

5. *If $x_1=x_0$ then $u_\#$ is the automorphism of $\pi_1(X,x_0)$ given by conjugation by $[u]\in\pi_1(X,x_0)$.*

Proof. We first check that $u_\#$ is a group homomorphism. $u_\#$ is well-defined, since homotopies respect function composition. Suppose that

γ_0 and γ_1 are two loops based at x_0. We have

$$\begin{aligned}
u_\#([\gamma_0 \cdot \gamma_1]) &= [u^{-1} \cdot \gamma_0 \cdot \gamma_1 \cdot u] \\
&= [u^{-1} \cdot \gamma_0 \cdot (\mathrm{id}) \cdot \gamma_1 \cdot u] \\
&= [u^{-1} \cdot \gamma_0 \cdot (u \cdot u^{-1}) \cdot \gamma_1 \cdot u] \\
&= [(u^{-1} \cdot \gamma_0 \cdot u) \cdot (u^{-1} \cdot \gamma_1 \cdot u)] \\
&= [(u^{-1} \cdot \gamma_0 \cdot u)][(u^{-1} \cdot \gamma_1 \cdot u)] \\
&= u_\#([\gamma_0])u_\#([\gamma_1]).
\end{aligned}$$

Thus $u_\#$ is a group homomorphism.

(1) follows as homotopies respect function composition. (2) is clear and (3) is follows as $u_\#$ is given by composition, and composition of functions is associative.

From (2) and (3) we have

$$\begin{aligned}
(u^{-1})_\# \circ u_\# &= (u \cdot u^{-1})_\# \\
&= (c_{x_0})_\# \\
&= \mathrm{id}_{\pi_1(X,x_0)}.
\end{aligned}$$

Similarly the other way around. It follows that

$$(u^{-1})_\# = (u_\#)^{-1}.$$

In particular $u_\#$ is an isomorphism. (5) just follows from unwrapping the definitions.

For (4) we have to prove that

$$(f \circ u)_\# \circ f_* = f_* \circ u_\#.$$

We calculate

$$\begin{aligned}
((f \circ u)_\# \circ f_*)([\gamma]) &= ((f \circ u)_\#)([f \circ \gamma]) \\
&= [(f \circ u)^{-1} \cdot (f \circ \gamma) \cdot (f \circ u)] \\
&= [(f \circ u^{-1}) \cdot (f \circ \gamma) \cdot (f \circ u)] \\
&= [f \circ (u^{-1} \cdot \gamma \cdot u)] \\
&= f_*[u^{-1} \cdot \gamma \cdot u] \\
&= f_*(u_\#[\gamma]) \\
&= (f_* \circ u_\#)([\gamma]).
\end{aligned}$$

This gives (4). □

Note we are now in a similar situation for the fundamental group as we were for homology. We would really like to understand topological spaces up to homeomorphism (or even homotopy equivalence). But the fundamental group depends on the choice of base point and even if X is path connected, the fundamental group of X is only defined up to isomorphism.