17. Paths and Loops

Definition 17.1. *Let X be a topological space.*

A ***path*** *γ from x_0 to x_1, denoted $x_0 \rightsquigarrow x_1$, is a continuous function*

$$\gamma\colon I \longrightarrow X$$

such that

$$\gamma(0) = x_0 \qquad \text{and} \qquad \gamma(1) = x_1.$$

A ***loop*** *is a path such that $x_1 = x_0$.*

If γ_0 is a path from x_0 to x_1 and γ_1 is a path from x_1 to x_2 then we define the ***concatenation*** *of γ_0 and γ_1, denoted $\gamma_0 \cdot \gamma_1$, as the path*

$$\gamma_0 \cdot \gamma_1\colon I \longrightarrow X$$

given by

$$(\gamma_0 \cdot \gamma_1)(t) = \begin{cases} \gamma_0(t) & t \in [0, 1/2] \\ \gamma_1(t) & t \in [1/2, 1]. \end{cases}$$

If γ is a path from x_0 to x_1 then the ***inverse*** *of γ, denoted γ^{-1}, is the path*

$$\gamma^{-1}\colon I \longrightarrow X \qquad \text{given by} \qquad \gamma^{-1}(t) = \gamma(1-t).$$

The constant path at x_0, denoted c_{x_0}, is

$$c_{x_0}\colon I \longrightarrow X \qquad \text{given by} \qquad c_{x_0}(t) = x_0.$$

As in Math 190A, the relation $\sim$ on X, given by

$$x \sim y \qquad \text{there is a path from } x \text{ to } y$$

is an equivalence relation. The equivalence classes are the connected components.

Definition-Lemma 17.2. *There is a functor π_0 from the category of topological spaces to the category of sets.*

To every topological space X, $\pi_0(X)$ is the set of equivalence classes of the relation $\sim$. To every continuous function $f\colon X \longrightarrow Y$ we define the function

$$\pi_0(f)\colon \pi_0(X) \longrightarrow \pi_0(Y) \qquad \text{by the rule} \qquad \pi_0(f)[x] = [f(x)].$$

Moreover the function $\pi_0(f)$ only depends the homotopy class of f.

Proof. We first have to check that $\pi_0(f)$ is well-defined. But this follows from the fact that the image of a path-connected space is path-connected. It is very easy to see that π_0 is a functor.

Now suppose that $f \sim g$. Then there is a continuous function

$$F\colon X \times I \longrightarrow X$$

such that

$$F(x,0) = f(x) \qquad \text{and} \qquad F(x,1) = g(x).$$

Define a path γ

$$\gamma\colon I \longrightarrow X \qquad \text{by the rule} \qquad \gamma(t) = F(x,t).$$

Then γ is a path from $f(x)$ to $g(x)$. Thus

$$\begin{aligned} \pi_0(f)[x] &= [f(x)] \\ &= [g(x)] \\ &= \pi_0(g)[x]. \end{aligned}$$

It follows that $\pi_0(f) = \pi_0(g)$. □

Definition 17.3. *Let X and Y be two topological spaces and let $A \subset X$ be a subset of X. We say that two continuous functions f and $g\colon X \longrightarrow Y$ are* ***homotopic relative to*** *A, if the there is a homotopy F from f to g such that*

$$F(a,t) = f(a) \qquad \textit{for every } a \in A.$$

Note that if f and g are homotopic relative to A then $f|_A = g|_A$.

Definition 17.4. *Let X be a topological space and let γ_0 and γ_1 be two paths from x_0 to x_1.*

Then γ_0 and γ_1 are ***homotopic as paths*** *if there a homotopy F from γ_0 to γ_1 relative to $A = \{0,1\} \subset I$.*

Lemma 17.5. *Let X be a topological space.*

If $\alpha_0 \sim \alpha_1$ as paths from x_0 to x_1 and $\beta_0 \sim \beta_1$ as paths from x_1 to x_2 then $\gamma_0 = \alpha_0 \cdot \beta_0 \sim \alpha_1 \cdot \beta_1 = \gamma_1$ as paths from x_0 to x_2.

Proof. Let A be a homotopy from α_0 to α_1 and let B be a homotopy from β_0 to β_1.

Define a function

$$C\colon I^2 \longrightarrow X$$

by the rule

$$C(s,t) = \begin{cases} A(2s,t) & s \in [0,1/2] \\ B(2s-1,t) & s \in [1/2,1]. \end{cases}$$

Then C is continuous by the patching lemma and

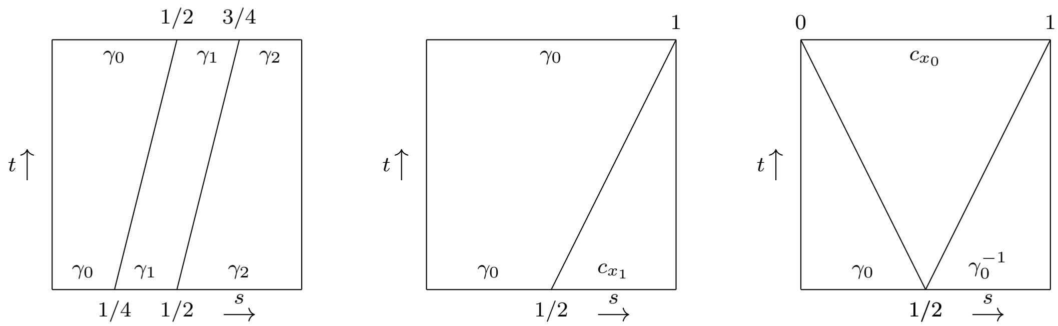
$$C(s,0) = \gamma_0(s) \quad C(s,1) = \gamma_1(s) \quad C(0,t) = x_0 \quad \text{and} \quad C(1,t) = x_2.$$

Thus C is a homotopy from γ_0 to γ_1 fixing x_0 and x_1. □

Lemma 17.6. *If X is topological space and γ_i are three paths $x_i \rightsquigarrow x_{i+1}$, $i = 0$, 1 and 2, then*

(1) $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \sim \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$*, as paths from* x_0 *to* x_3*,*
(2) $c_{x_0} \cdot \gamma_0 \sim \gamma_0 \sim \gamma_0 \cdot c_{x_1}$ *as paths from* x_0 *to* x_1*, and*
(3) $\gamma_0 \cdot \gamma_0^{-1} \sim c_{x_0}$ *as paths from* x_0 *to* x_0 *and* $\gamma_0^{-1} \cdot \gamma_0 \sim c_{x_1}$ *as paths from* x_1 *to* x_1*.*

Proof. The proofs of these results are built on the following three diagrams.



We first prove (i). Define a function

$$F\colon I^2 \longrightarrow X$$

by the rule

$$F(s,t) = \begin{cases} \gamma_0(\frac{4}{t+1}s) & 0 \leq s \leq \frac{t+1}{4} \\ \gamma_1(4s-1-t) & \frac{t+1}{4} \leq s \leq \frac{t+2}{4} \\ \gamma_2(1-\frac{4(1-s)}{2-t}) & \frac{t+2}{4} \leq s \leq 1. \end{cases}$$

Then F is continuous by the patching lemma as the pieces are continuous and they agree on overlaps. Looking at the picture F is a homotopy from $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2$ to $\gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$, as paths from x_0 to x_3. Now we turn to (ii). Define a function

$$F\colon I^2 \longrightarrow X$$

by the rule

$$F(s,t) = \begin{cases} \gamma_0(\frac{2}{t+1}s) & 0 \leq s \leq \frac{t+1}{2} \\ x_1 & \frac{t+1}{2} \leq s \leq 1. \end{cases}$$

F is a homotopy from $\gamma_0 \cdot c_{x_1}$ to γ_0 as paths from x_0 to x_1.

For (iii) define a function

$$F\colon I^2 \longrightarrow X$$

by the rule

$$F(s,t) = \begin{cases} \gamma_0(2s) & 0 \leq s \leq \frac{1-t}{2} \\ \gamma_0(1-t) & \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \gamma_0(2-2s) & \frac{1+t}{2} \leq s \leq 1. \end{cases}$$

F is a homotopy from $\gamma_0 \cdot \gamma_0^{-1}$ to c_{x_0} as paths from x_0 to x_0. $\square$