16. Homotopy Equivalence

Definition 16.1. *A continuous function $f\colon X \longrightarrow Y$ between two topological spaces is a* ***homotopy equivalence*** *if there is a continuous function $g\colon Y \longrightarrow X$ such that*

$$g \circ f \sim id_X \colon X \longrightarrow X \qquad \text{and} \qquad f \circ g \sim id_Y \colon Y \longrightarrow Y.$$

We call g a ***homotopy inverse*** *of f. If there is a homotopy equivalence from X to Y then we say that X and Y are* ***homotopy equivalent****.*

Example 16.2. *Let $X = S^1$ and $Y = \mathbb{R}^2 \setminus \{ 0 \}$.*

Let

$$i\colon S^1 \longrightarrow \mathbb{R}^2 \setminus \{ 0 \}$$

be the natural inclusion map and define a function

$$r\colon \mathbb{R}^2 \setminus \{ 0 \} \longrightarrow S^1 \qquad \text{by the rule} \qquad x \longrightarrow \frac{x}{||x||}.$$

Note that

$$r \circ i = \mathrm{id}_{S^1}.$$

Thus $r \circ i$ is certainly chain homotopic to the identity.

Define the function

$$F\colon (\mathbb{R}^2 \setminus \{ 0 \}) \times I \longrightarrow \mathbb{R}^2 \setminus \{ 0 \} \qquad \text{by the rule} \qquad (x,t) \longrightarrow \frac{x}{t + (1-t)||x||}.$$

Note that F is continuous as it is the composition of the continuous map

$$(x,t) \longrightarrow (x, t + (1-t)||x||)$$

to $\mathbb{R}^2 \setminus \{ 0 \}$ and r. When $t = 0$ the above map is the identity and so

$$F(x,0) = r(x)$$

but when $t = 1$ then the above map is the constant map and we get

$$F(x,1) = \mathrm{id}_Y(x).$$

Thus S^1 is homotopy equivalent to $\mathbb{R}^2 \setminus \{ 0 \}$.

Example 16.3. *Let $X = \{ 0 \}$ and $Y = \mathbb{R}^n$.*

Let

$$i\colon \{ 0 \} \longrightarrow \mathbb{R}^n$$

be the natural inclusion map and let r be the constant map

$$r\colon \mathbb{R}^n \longrightarrow \{ 0 \} \qquad \text{given by} \qquad x \longrightarrow 0.$$

Note that

$$r \circ i = \mathrm{id}_X.$$

Thus $r \circ i$ is certainly chain homotopic to the identity.

Define the function

$$F\colon \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n \qquad \text{by the rule} \qquad (x,t) \longrightarrow tx.$$

F is certainly continuous. When $t = 0$

$$F(x,0) = 0 = (i \circ r)(x).$$

but when $t = 1$ then

$$F(x,1) = \mathrm{id}_Y(x).$$

Thus S^1 is homotopy equivalent to a point.

Definition 16.4. *A topological space is* ***contractible*** *if it is homotopy equivalent to the one point space* $\{*\}$.

$\mathbb{R}^n$ is contractible to a point.

Lemma 16.5. *Let X, Y and Z be three topological spaces. If*

$$f_i\colon X \longrightarrow Y \qquad \textit{for} \qquad i = 0,1$$

and

$$g_i\colon Y \longrightarrow Z \qquad \textit{for} \qquad i = 0,1$$

are homotopic maps then

$$h_i = g_i \circ f_i\colon X \longrightarrow Z \qquad \textit{for} \qquad i = 0,1$$

are homotopic maps.

Proof. Let $g = g_0$. We first show that

$$k_i = g \circ f_i\colon X \longrightarrow Z \qquad \text{for} \qquad i = 0,1$$

are homotopic maps. By assumption there is a homotopy

$$F\colon X \times I \longrightarrow Y$$

from f_0 to f_1. Let

$$K\colon X \times I \longrightarrow Z$$

be the composition, $K = g \circ F$. Then K is continuous and

$$K(x,0) = g \circ f_0(x) = h_0(x) \qquad \text{and} \qquad K(x,1) = g \circ f_1(x) = h_1(x).$$

Thus $k_0 \sim k_1$. A similar argument shows that

$$g_0 \circ f \sim g_1 \sim f,$$

where $f = f_1$. □

Lemma 16.6. *The relation $\sim$ on topological spaces given by $X \sim Y$ if and only if X and Y are homotopy equivalent is an equivalence relation.*

Proof. We have to check that $\sim$ is reflexive, symmetric and transitive.
Let X be a topological space. Then the identity

$$\mathrm{id}_X \colon X \longrightarrow X$$

is a homotopy equivalence, since

$$\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$$

is chain homotopic to the identity. Thus $\sim$ is reflexive.
Now suppose that X and Y are two topological spaces and $X \sim Y$. Then we can find continuous functions

$$f \colon X \longrightarrow Y \qquad \text{and} \qquad g \colon Y \longrightarrow X$$

such that $g \circ f$ and $f \circ g$ are homotopic to the identity.
In this case

$$g \colon Y \longrightarrow X \qquad \text{and} \qquad f \colon X \longrightarrow Y$$

are continuous maps such that such that $f \circ g$ and $g \circ f$ are homotopic to the identity. Thus $Y \sim X$. Thus $\sim$ is symmetric.
Now suppose that we are given three topological spaces X, Y and Z and $X \sim Y$ and $Y \sim Z$. In this case we are given continuous maps

$$f \colon X \longrightarrow Y \qquad \text{and} \qquad f' \colon Y \longrightarrow X$$

and

$$g \colon Y \longrightarrow Z \qquad \text{and} \qquad g' \colon Z \longrightarrow Y$$

such that f and f' are homotopy inverses and g and g' are homotopy inverses.
It follows that

$$\begin{aligned} (f' \circ g') \circ (g \circ f) &= f' \circ (g' \circ g) \circ f \\ &\sim f' \circ (\mathrm{id}_Y \circ f) \\ &= f' \circ f \\ &\sim \mathrm{id}_X. \end{aligned}$$

Similarly the other way around

$$(g \circ f) \circ (f' \circ g') \sim \mathrm{id}_Z.$$

Thus $g \circ f$ and $f' \circ g'$ are homotopy inverses. It follows that $X \sim Z$. Thus $\sim$ is transitive. □

Definition 16.7. *Let X be a topological space, let A be a subspace and let $i \colon A \longrightarrow X$ be the natural inclusion*

We say that A is a ***retraction*** *of X if there is a continuous function $r \colon X \longrightarrow A$ such that $r \circ i = id_A$.*

*We say that A is a **deformation retraction** if in addition* $i \circ r \sim id_X$.