15. Homotopies

We now explore a different way to associate algebraic invariants to a topological space X. The basic idea is to use loops. We fix a point x_0 of X (called the **base point**) and consider continuous maps

$$I \longrightarrow X$$

which send both 0 and 1 to x_0. We want to get a group and so the obvious thing is to compose (or perhaps better concatenate) loops. The problem is that we really get a map from $[0, 2]$ to X. We could shrink $[0, 1]$ down to $[0, 1/2]$ and concatenate this with a loop from $[1/2, 1]$. But then what about associativity?

More seriously there are just an unimaginable number of loops to any interesting topological space. To get a reasonable size group, we consider two loops to be the same if one can continuously deform one loop to another.

All of this leads to the notion of

Definition-Lemma 15.1. *Let X and Y be two topological spaces and let*

$$f \colon X \longrightarrow Y \qquad \text{and} \qquad g \colon X \longrightarrow Y$$

be two continuous functions.

*A **homotopy** between f an g is a continuous function*

$$F \colon X \times I \longrightarrow Y$$

such that

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(x, 1) = g(x).$$

In this case we say that f and g are homotopic maps.

If we define a relation $\sim$ on continuous functions from X to Y by the rule $f \sim g$ if and only if f and g are homotopic then we get an equivalence relation.

Proof. We have to check that $\sim$ is reflexive, symmetric and transitive.

Suppose that f is a continuous function from X to Y. Define a function

$$F \colon X \times I \longrightarrow Y$$

by the rule

$$F(x, t) = f(x).$$

Then

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(x, 1) = f(x)$$

so that $f \sim f$ and $\sim$ is reflexive.

Now suppose that f and g are two continuous functions from X to Y and $f \sim g$. Then we may find

$$F\colon X \times I \longrightarrow Y$$

such that

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(x, 1) = g(x).$$

Let

$$G\colon X \times I \longrightarrow Y$$

be the function

$$G(x, t) = F(x, 1 - t).$$

Then G is a continuous function as it is the composition of F with the continuous function

$$X \times I \longrightarrow X \times I \qquad \text{given by} \qquad (x, t) \longrightarrow (x, 1 - t).$$

We have

$$\begin{aligned} G(x, 0) &= F(x, 1) \\ &= g(x) \end{aligned}$$

and

$$\begin{aligned} G(x, 1) &= F(x, 0) \\ &= f(x). \end{aligned}$$

Thus $g \sim f$ and $\sim$ is symmetric.

Now suppose that f, g and h are three continuous functions from X to Y and $f \sim g$ and $g \sim h$. In this case there are two continuous functions

$$F\colon X \times I \longrightarrow Y \qquad \text{and} \qquad G\colon X \times I \longrightarrow Y$$

such that

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(x, 1) = g(x).$$

and

$$G(x, 0) = g(x) \qquad \text{and} \qquad G(x, 1) = h(x).$$

Define a function

$$H\colon X \times I \longrightarrow Y$$

by the rule

$$H(x, t) = \begin{cases} F(x, 2t) & \text{for } t \in [0, 1/2] \\ G(x, 2t - 1) & \text{for } t \in [1/2, 1]. \end{cases}$$

It is easy to see that $F(x, 2t)$ and $G(x, 2t-1)$ are continuous functions. The two sets $[0, 1/2]$ and $[1/2, 1]$ are closed and the two functions agree on the overlap, since

$$F(x, 1) = g(x) = G(x, 0).$$

On the other hand

$$\begin{aligned} H(x, 0) &= F(x, 0) \\ &= f(x), \end{aligned}$$

and

$$\begin{aligned} H(x, 1) &= G(x, 1) \\ &= h(x), \end{aligned}$$

so that $f \sim h$. Thus $\sim$ is transitive and so it is an equivalence relation. □

We will spend much of the rest of this class trying to understand the equivalence classes.