## 14. The pushforward

Our goal is to prove (13.4) so that we can define the pushforward.

*Proof of* (13.4). Given $\sigma \in \mathcal{K}$, we let $\mathcal{K}(\sigma)$ be the subcomplex defined by $\sigma$. We already know the reduced homology is zero. Let $\mathcal{L}(\sigma)$ be the subcomplex of $\mathcal{L}$ with support $\sigma$. Note that if $\sigma_0$ is a face of $\sigma$ then

$$C_\bullet(\mathcal{K}(\sigma_0)) \subset C_\bullet(\mathcal{K}(\sigma)) \qquad \text{and} \qquad C_\bullet(\mathcal{L}(\sigma_0)) \subset C_\bullet(\mathcal{L}(\sigma)).$$

Given a simplex $\tau \in \mathcal{L}$ we let $\sigma_\tau$ be the smallest simplex in $\mathcal{K}$ containing $\tau$. Note that if $\tau_0$ is a face of $\tau$ then $\sigma_{\tau_0}$ is a face of $\sigma_\tau$. Thus

$$C_\bullet(\mathcal{L}(\sigma_{\tau_0})) \subset C_\bullet(\mathcal{L}(\sigma_\tau)).$$

**Step I** We prove this result under the assumption that the reduced homology of $\mathcal{L}(\sigma)$ is zero.

Let $\lambda$ be the chain map defined by the condition that

$$\lambda(\sigma) \in C_\bullet(\mathcal{L}(\sigma)),$$

(13.5). Note that

$$(i_* \circ \lambda)(\sigma) \in C_\bullet(K(\sigma)).$$

The same is true of the identity map. Thus (13.5) implies that $i_* \circ \lambda$ is chain homotopic to the identity.

Note that

$$i_*(\tau) \in C_\bullet(\mathcal{K}(\sigma_\tau)).$$

It follows that

$$(\lambda \circ i_*)(\tau) \in C_\bullet(\mathcal{L}(\sigma_\tau)).$$

The same is true of the identity map. Thus (13.5) implies that $\lambda \circ i_*$ is chain homotopic to the identity.

It follows that $i_*$ is an isomorphism on homology and that $\lambda_*$ is the inverse of $i_*$.

From now on we may assume that $\mathcal{K}$ is simply the complex given by a simplex $\sigma$ and all its faces and using Step I all we need to show is the any subdvision $\mathcal{L}$ of $\mathcal{K}$ has reduced homology zero.

**Step II** We show that the result holds if $\mathcal{L}$ is the $r$th Barycentric subdvision of $\mathcal{K}$.

By induction we may assume that $r = 1$. In this case the simplices of $\mathcal{L}$ are cones over the faces of simplices in the boundary of $\sigma$. But then the reduced homology of $\mathcal{L}$ is zero and we may use Step I.

**Step III** Note that $i$ is a simplicial approximation of the identity. Pick $r$ so that the identity has a simplicial approximation from $\mathcal{L}^{(r)}$ to $\mathcal{K}$. Similarly pick $s$ so that the identity has a simplicial approximation $g$ from $\mathcal{K}^{(s)}$ to $\mathcal{L}^{(r)}$.

Then $i \circ f$ is a simplicial approximation of the identity, so that $i_* \circ f_*$ is the identity on homology. Thus $f_*$ is injective on homology. On the other hand $f \circ g$ is is a simplicial approximation of the identity, so that $f_* \circ g_*$ is the identity on homology. Thus $f_*$ is surjective on homology.

It follows that $f_*$ is an isomorphism. But then $i_*$ is an isomorphism. But then the reduced homology is zero. □

**Definition-Theorem 14.1.** *Let*

$$h \colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

*be a continuous map of simplicial complexes.*

*Then there is a* ***pushforward*** $h_*$ *on homology,*

$$h_* \colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}),$$

*that is natural in the following sense. If* $h$ *is the identity then* $h_*$ *is the identity and if we are given three simplicial complexes* $\mathcal{K}$*,* $\mathcal{L}$ *and* $\mathcal{M}$ *and two continuous maps*

$$f \colon |\mathcal{K}| \longrightarrow |\mathcal{L}| \qquad \text{and} \qquad g \colon |\mathcal{L}| \longrightarrow |\mathcal{M}|$$

*then*

$$(g \circ f)_* = g_* \circ f_*.$$

*Proof.* Suppose that we are given a subdvision $\mathcal{M}$ of $\mathcal{K}$ and a simplicial approximation $f$ of $h$. We can then define

$$h_* = f_* \circ \lambda_* \colon H_m(\mathcal{K}) \longrightarrow H_m(\mathcal{L}).$$

Note that by (12.3) we can always take $\mathcal{M} = \mathcal{K}^{(r)}$, for some $r > 0$.

Now we have to check that $h_*$ does not depend on any choices.

Suppose that we are given two subdvisions $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{K}$ and two simplicial approximations $f_1$ and $f_2$ of $h$. Pick a third simplicial simplicial approximation $\mathcal{M}_3$ such that the identity map has two simplicial approximations $g_1$ and $g_2$, from $\mathcal{M}_3$ to $\mathcal{M}_1$ and from $\mathcal{M}_3$ to $\mathcal{M}_2$.

There are now two obvious ways to pushforward using $f_1$, directly as

$$f_{1*} \circ \lambda_*$$

or indirectly using

$$\begin{aligned}(f_1 \circ g_1)_* \circ (g_1^{-1})_* \circ \lambda_* &= f_{1*} \circ (g_{1*} \circ g_{1*}^{-1}) \circ \lambda_* \\ &= f_{1*} \circ \lambda_*.\end{aligned}$$

Here used the fact that subdvision operator from $\mathcal{M}_3$ to $\mathcal{M}_2$ is simply the inverse of $g_{1*}$.

Since these come out the same, using $\mathcal{M}_1$ is the same as using $\mathcal{M}_3$, to define the pushforward. By the same token using $\mathcal{M}_2$ is the same as using $\mathcal{M}_3$, to define the pushforward. Thus using $\mathcal{M}_1$ is the same as using $\mathcal{M}_2$, to define the pushforward.

The two properties of the pushforward are now clear. $\square$

Now suppose that $X$ is a triangulable topological space. Recall that this means are given a pair $(\mathcal{K}, f)$ of a simplicial complex and a homeomorphism

$$f\colon |\mathcal{K}| \longrightarrow X$$

Using this we can define the homology of $X$ as the homology of $\mathcal{K}$,

$$H_m(X) = H_m(\mathcal{K}).$$

This hides one unfortunate subtlety. It is true that the homology of $\mathcal{K}$ is well-defined and we have a well-defined pushforward.

However if we choose a different triangulation then we don't get the same homology groups. We do get isomorphic homology groups. Put differently there is a pushforward on the homology of $X$ buts this is only well-defined up to choice of triangulations.

There is one place where we do get a well-defined pushforward, that does not involve any choices. Suppose that we are given an endomorphism of $X$, a continuous map

$$h\colon X \longrightarrow X.$$

Then we can choose the same triangulation for both the domain and the range. This does give us a well-defined pushforward.

This might make more sense in a concrete example. Take $X = S^1$. There are two types of isomorphism $f\colon S^1 \longrightarrow S^1$. Indeed, let $\mathcal{K}$ be the 1-skeleton of a 2-simplex. Then $|\mathcal{K}|$ is homeomorphic to $S^1$. But there are two types of simplicial maps from $\mathcal{K}$ to $\mathcal{K}$. Ones that preserve orientation and those that don't.