13. Chain complexes

Our goal is to define the pushforward on homology for an arbitrary continuous map of simplicial complexes. The only thing that remains is to show that the homology does not change if we pass to a subdvision.

First an easy observation:

Lemma 13.1. *Let $\mathcal{L}$ be a simplicial subdvision of $\mathcal{K}$.*

Then the natural inclusion i of $\mathcal{L}$ in $\mathcal{K}$ is a simplicial map that satisfies the star condition. In particular it is a simplicial approximation to the identity.

Proof. The natural inclusion is simply the identity on the support. In particular it is easy to see that it is a simplicial map.

If $w \in \mathcal{L}$ is a vertex of $\mathcal{L}$ then w is in the interior of a unique simplex σ of $\mathcal{K}$. Let v be a vertex of σ. We will show that

$$\operatorname{St}(w, \mathcal{L}) \subset \operatorname{St}(v, \mathcal{K}).$$

If τ is a simplex that contains w then τ is contained in σ. But then the interior of τ is contained in the interior of σ. □

We want to show that i_* is an isomorphism on homology. The best thing to would be to write down a simplicial map from $\mathcal{K}$ to $\mathcal{L}$ and show that the induced map on homology is the inverse of i_*. However there is no such map.

The next best thing is to define something at the level of chains. It is convenient to introduce a little bit of notation.

Definition 13.2. *A* ***chain complex*** $C_\bullet$ *is a sequence of abelian groups $C_1, C_2, \dots$ and group homomorphisms*

$$\partial_m \colon C_m \longrightarrow C_{m-1}$$

such that $\partial_{m-1} \circ \partial_m = 0$.

A ***chain map*** $f_\bullet$ *between two chain complexes $C_\bullet$ and $D_\bullet$ is a sequence of group homomorphisms*

$$f_m \colon C_m \longrightarrow D_m$$

such that the following diagram commutes

$$\begin{array}{ccc} C_m & \xrightarrow{\partial_m} & C_{m-1} \\ {\scriptstyle f_m}\big\downarrow & & \big\downarrow{\scriptstyle f_{m-1}} \\ D_m & \xrightarrow{\partial_m} & D_{m-1}. \end{array}$$

In fact this defines a category. Its objects are chain complexes and its morphisms are chain maps. It is easy to check that the identity map on each C_m acts as the identity, it is clear how to compose chain maps, it is not hard to check that the composition is a chain map and associativity is straightforward.

For us the most interesting examples of chain complexes both come from a simplicial complex $\mathcal{K}$. The usual chain complex and the augmented chain complex.

Definition-Lemma 13.3. *Let $C_\bullet$ be a chain complex. The **boundaries** and **cycles** $Z_\bullet$ are the sequence of groups*

$$B_m = \operatorname{Im} \partial_m \qquad \text{and} \qquad Z_m = \operatorname{Ker} \partial_m.$$

*Every boundary is a cycle. The **homology** $H_\bullet$ of the complex is the sequence of groups*

$$H_m = \frac{Z_m}{B_m}.$$

A chain map $f_\bullet$ from $C_\bullet$ to $D_\bullet$ gives rise to a sequence of group homomorphisms

$$f_{m*} \colon H_m(C_\bullet) \longrightarrow H_m(D_\bullet).$$

Proof. This is similar to the argument for simplicial homology. □

Note that chain homotopies make sense between two chain complexes and a chain homotopy between two chain maps implies they induce the same map on homology.

Definition-Theorem 13.4. *Let $\mathcal{L}$ be a subdvision of $\mathcal{K}$.*

Then there is a unique chain map $\lambda_\bullet$ from $C_\bullet(\mathcal{K})$ to $C_\bullet(\mathcal{L})$, a sequence of group homomorphisms

$$\lambda = \lambda_m \colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L})$$

*called the **subdivision operator** such that*

$$i_* \circ \lambda = id \colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{K})$$

and

$$\lambda \circ i_* \colon C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{L})$$

is chain homotopic to the identity.

In particular λ_* is the inverse of i_* on homology, so that both i_* and λ are isomorphisms.

We will need a preparatory:

Lemma 13.5. *Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes.*

We suppose that for every simplex σ in $\mathcal{K}$ we are given a subcomplex $\mathcal{L}(\sigma) \subset \mathcal{L}$ whose reduced homology is zero, such that if $\tau < \sigma$ then $\mathcal{L}(\tau) \subset \mathcal{L}(\sigma)$.

(1) If f and g are chain maps from $C_\bullet(\mathcal{K})$ to $C_\bullet(\mathcal{L})$ that send vertices to vertices and the cycles $f(\sigma)$ and $g(\sigma)$ both belong to $C_\bullet(\mathcal{L}(\sigma))$ then f and g are chain homotopic.

(2) There is a chain map λ from $C_\bullet(\mathcal{K})$ to $C_\bullet(\mathcal{L})$ that send vertices to vertices and $\lambda(\sigma) \in C_\bullet(\mathcal{L}(\sigma))$.

Proof. The proof of (1) is a small variation on the proof that if f and g are contiguous then they are chain homotopic.

We now turn to (2). We want to define a group

$$\lambda = \lambda_m \colon C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L}),$$

such that $\lambda_{m-1} \circ \partial_m = \partial_{m-1} \circ \lambda_m$. It is enough to construct such a λ on simplices and extend by linearity. We proceed by induction on m.

Given v pick a vertex $u \in \mathcal{L}(v)$ and send v to u, $\lambda(v) = u$. It is clear that

$$\partial_0 \lambda(v) = \lambda(\partial_0 v),$$

since both sides are zero.

Now suppose that

$$\sigma = \langle v, w \rangle \in \mathcal{K}$$

is a 1-simplex. Then

$$\lambda_0(\partial_1 \sigma) \in C_0(\mathcal{L}(\sigma))$$

has degree zero. Thus it is homologous to zero and so we may find a 1-chain $\beta \in C_1(\mathcal{L}(\sigma)$ such that

$$\partial_1 \beta = \lambda(\partial_1 \sigma).$$

We define

$$\lambda_1(\sigma) = \beta.$$

Extending by linearity, this defines λ on 1-chains.

Suppose that λ has been defined on p-chains, for all $p < m$. Pick an m-simplex σ. Then

$$\gamma = \lambda_{m-1}(\partial_m \sigma) \in C_{m-1}(\mathcal{L}(\sigma)).$$

Note that $\partial_m \sigma$ is a sum over the faces τ of σ. λ_{m-1} applied to the face τ lands in $C_{m-1}(\mathcal{L}(\tau)$, which sits inside $C_{m-1}(\mathcal{L}(\sigma))$. Then γ does lie in $C_{m-1}(\mathcal{L}(\sigma))$.

We have

$$\begin{aligned}
\partial_{m-1}\gamma &= \partial_{m-1}(\lambda_{m-1}(\partial_m\sigma)) \\
&= (\partial_{m-1}\lambda_{m-1})(\partial_m\sigma) \\
&= (\lambda_{m-2}\partial_{m-1})(\partial_m\sigma)) \\
&= \lambda_{m-2}(\partial_{m-1}\partial_m\sigma) \\
&= \lambda_{m-2}(0) \\
&= 0.
\end{aligned}$$

By assumption the reduced homology of $\mathcal{L}(\sigma)$ is zero. Thus we may find $\beta \in C_m(\mathcal{L}(\sigma))$ such that

$$\partial_m\beta = \gamma.$$

We defined

$$\lambda(\sigma) = \gamma.$$

Extending by linearity, this defines λ on m-chains and this completes the induction. □