12. Simplicial Approximation Theorem

Let $\mathcal{K}$ be a simplicial complex. Then

$$|\mathcal{K}| \subset \mathbb{R}^N$$

some N. In particular the norm on $\mathbb{R}^N$ induces a metric on $|\mathcal{K}|$.

Definition 12.1. *Let $\mathcal{K}$ be a simplicial complex.*
*The **mesh** of $\mathcal{K}$ is*

$$\mu(\mathcal{K}) = \{\, ||v_1 - v_0|| \,|\, \langle v_0, v_1 \rangle \in \mathcal{K} \,\}.$$

In words the mesh of $\mathcal{K}$ is the greatest distance between two vertices that are the endpoints of an edge.

Definition-Theorem 12.2. *Let $\mathcal{K}$ be a simplicial complex.*
*The rth **Barycentric subdvision** is defined as follows.*

$$K^{(r)} = \begin{cases} \mathcal{K} & \text{if } r = 0 \\ \mathcal{L} & \text{where } \mathcal{L} \text{ is the first Barycentric subdvision of } \mathcal{K}^{(r-1)}. \end{cases}$$

If $\mathcal{K}$ has dimension at most n then

$$\mu(K^{(r)}) \leq \left(\frac{n}{n+1} \right)^r \mu(\mathcal{K}).$$

In particular

$$\lim_{r \to \infty} \mu(\mathcal{K}) = 0.$$

Proof. By an obvious induction, it suffices to do the case $r = 1$. Suppose that

$$\langle \hat{\tau}, \hat{\sigma} \rangle \in \mathcal{L}.$$

Then $\tau < \sigma$ are simplices in $\mathcal{K}$. Note that the distance of $\hat{\sigma}$ to points of σ is maximised by one of the vertices so that

$$||\hat{\sigma} - \hat{\tau}|| \leq \max\{\, ||\hat{\sigma} - v|| \,|\, v \text{ is a vertex of } \sigma. \}$$

Thus there is no harm in assuming that τ is a vertex of σ. Suppose that

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle.$$

We may also assume that $\tau = v_0$. Now we calculate

$$\begin{aligned} ||\hat{\sigma} - v_0|| &= ||\frac{1}{m+1}\sum_{i=0}^{m} v_i - v_0|| \\ &= ||\frac{1}{m+1}\sum_{i=0}^{m} v_i - \frac{m+1}{m+1}v_0|| \\ &= \frac{1}{m+1}||\sum_{i=0}^{m}(v_i - v_0)|| \\ &\leq \frac{1}{m+1}\sum_{i=1}^{m}||v_i - v_0|| \\ &\leq \frac{m}{m+1}\mu(\sigma) \\ &\leq \frac{m}{m+1}\mu(\mathcal{K}) \\ &\leq \frac{n}{n+1}\mu(\mathcal{K}). \end{aligned}$$

□

Theorem 12.3 (Simplicial approximation theorem)**.** *Let*

$$h\colon \mathcal{K} \longrightarrow \mathcal{L}$$

be a continuous map of simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.

Then there is an integer r and a simplicial approximation f of h, from $\mathcal{K}^{(r)}$ to $\mathcal{L}$.

Proof. Consider the open cover

$$\mathcal{U} = \{\, h^{-1}(\mathrm{St}(w, \mathcal{L})) \mid w \text{ is a vertex of } \mathcal{L} \,\}.$$

This has a Lebesgue number δ. Thus for every point x of $|\mathcal{K}|$ the open ball of radius δ about x is entirely contained in one of the open subsets in $\mathcal{U}$.

(12.2) implies that we may choose r so that

$$\mu(\mathcal{K}^{(r)}) < \delta.$$

For each vertex v we have

$$\begin{aligned} \mathrm{St}(v, \mathcal{K}^{(r)}) &\subset B_\delta(v) \\ &\subset h^{-1}(\mathrm{St}(w, \mathcal{L})), \end{aligned}$$

for some vertex w of $\mathcal{L}$. In other words,

$$h(\mathrm{St}(v, \mathcal{K}^{(r)})) \subset \mathrm{St}(w, \mathcal{L}).$$

But then h has a simplicial approximation. □