11. Barycentric subdvision

We want to define a pushforward on homology for a continuous map h between triangulable spaces. We saw in §10 that this exists if h has a simplicial approximation.

It is not hard to write down triangulations of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ and a continuous map between them that does not have a simplicial approximation.

However there is an easy fix for this. We just need to make the triangulation of $\mathcal{K}$ fine enough so that the star condition holds. This has the form of an ϵ-δ argument. The triangulation on $\mathcal{L}$ gives us the ϵ and if we subdivide $\mathcal{K}$ finely enough then we can make δ sufficiently small.

Definition 11.1. *Let $\mathcal{K}$ be a simplicial complex.*

*A **subdivision** of $\mathcal{K}$ is a simplicial complex $\mathcal{L}$ such that every simplex in $\mathcal{L}$ is contained in a simplex of $\mathcal{K}$ and the support of $\mathcal{K}$ is the same as the support of $\mathcal{L}$,*

$$|\mathcal{K}| = |\mathcal{L}|.$$

Given any particular simplicial complex $\mathcal{K}$ it is not hard to write down an appropriate subdivision. However we need a procedure that works for any simplicial complex. It is not hard to figure out how to subdivide a simplex. The only small issue is to make sure we divide all of the simplices compatibly.

One way to arrange this is to use Barycentric subdivision. The basic idea is to put a vertex in the middle and subdivide accordingly. Here is what we mean by the middle:

Definition 11.2. *Let*

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle$$

*be a simplex. The **Barycentre** of σ is the point*

$$\hat{\sigma} = \frac{v_0 + v_1 + \cdots + v_m}{p+1}.$$

There are two ways to think of Barycentric subdivision. One way is as an interative process, that goes from the p-skeleton to the $(p+1)$-skeleton. At each stage we put in the Barycentre of a $(p+1)$-simplex and use this to take the cone over all the simplices we have constructed so far. This gives a nice intuitive picture of what is going on but after a while one starts to drown in notation.

The second way is simply to write down all of the simplices in the Barycentric subdivision at once. This does mean there is more to check all at once:

A little bit of notation. Given two simplices τ and σ, we write

$$\tau < \sigma$$

if τ is a proper face of σ.

Definition-Theorem 11.3. *Let $\mathcal{K}$ be a simplicial complex.*

The set

$$\mathcal{L} = \{\, \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m \rangle \,|\, \sigma_0 < \sigma_1 < \cdots < \sigma_m \text{ are simplices in } \mathcal{K} \,\}$$

*is a simplicial subdivision of $\mathcal{K}$, called the **(first) Barycentric subdivision** of $\mathcal{K}$.*

Proof. We first check that $\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m$ span an m-plane. Suppose not. Then we can find real numbers $\lambda_0, \lambda_1, \dots, \lambda_m$ such that

$$\sum \lambda_i \hat{\sigma}_i = 0 \qquad \text{and} \qquad \sum \lambda_i = 1.$$

Let j be the largest index such that $\lambda_j \neq 0$. Let

$$\mu_i = -\frac{\lambda_i}{\lambda_j}.$$

Solving for σ_j we get

$$\begin{aligned} \hat{\sigma}_j &= \sum_{i<j} -\frac{\lambda_i}{\lambda_j} \hat{\sigma}_i \\ &= \sum_{i<j} \mu_i \hat{\sigma}_i. \end{aligned}$$

Thus $\hat{\sigma}_j$ lies in the span of $\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_{j-1}$, that is, the space spanned by a proper face of σ_j. This is not possible. Hence $\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m$ span an m-plane.

Now we check that $\mathcal{L}$ is a simplicial complex. Suppose that

$$\Sigma = \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m \rangle \in \mathcal{L}$$

is an m-simplex in $\mathcal{L}$. To get a face of this simplex, we simply omit some of the simplices from the sequence

$$\sigma_0 < \sigma_1 < \cdots < \sigma_m.$$

This is then a simplex T in $\mathcal{L}$.

Now suppose that we are given two simplices in $\mathcal{L}$,

$$\Sigma = \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_p \rangle \qquad \text{and} \qquad T = \langle \hat{\tau}_0, \hat{\tau}_1, \dots, \hat{\tau}_q \rangle.$$

Note that

$$\hat{\sigma}_i \in \sigma_i \subset \sigma_p \qquad \text{for all} \qquad i \leq p$$

so that $\Sigma \subset \sigma_p$. Similarly $T \subset \tau_q$. Thus

$$\Sigma \cap T \subset \sigma_p \cap \tau_q.$$

But

$$\delta = \sigma_p \cap \tau_q$$

is a simplex in $\mathcal{K}$. Let i be the largest index so that $\hat{\sigma}_i \in \delta$. Then

$$\Sigma_1 = \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_i \rangle \in \mathcal{L}$$

since we just thinned out the original sequence

$$\sigma_0 < \sigma_1 < \dots < \sigma_m.$$

Note that

$$\Sigma_1 = \Sigma \cap \delta.$$

We can repeat a similar process for T to get a simplex $T_1 \in \mathcal{L}$ such that

$$T_1 = T \cap \delta.$$

Note that

$$\Sigma \cap T = \Sigma_1 \cap T_1.$$

Replacing Σ by Σ_1 and T by T_1, we may assume that Σ and T both belong to the same simpex $\delta \in \mathcal{K}$. Suppose that δ is an m-simplex. Let $\mathcal{M}$ be the $(m-1)$-skeleton of δ. Then $\mathcal{M}$ is a subcomplex of $\mathcal{K}$.

There are two cases. Suppose that Σ and T both contain $\hat{\delta}$. Then Σ and T are cones with vertex $\hat{\delta}$ over smaller dimensional simplices, Σ_1 and T_1. Σ_1 and T_1 lie in $\mathcal{M}$. By induction $P_1 = \Sigma_1 \cap T_1$ is a face of both Σ_1 and T_1. But then

$$P = \langle \hat{\delta}, P_1 \rangle$$

is a face of both

$$\Sigma = \langle \hat{\delta}, \Sigma_1 \rangle \qquad \text{and} \qquad T = \langle \hat{\delta}, T_1 \rangle.$$

Now suppose that Σ and T do not both contain $\hat{\delta}$. Then $\Sigma \cap T$ lies in $\mathcal{M}$ and we are done by induction. It follows that $\mathcal{L}$ is a simplicial complex.

Now we show that $\mathcal{L}$ is a subdivision of $\mathcal{K}$. If

$$\Sigma = \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m \rangle \in \mathcal{L}$$

then

$$\Sigma \subset \sigma_m.$$

Now suppose that $x \in |\mathcal{K}|$. Then

$$x \in \sigma = \langle v_0, v_1, \dots, v_m \rangle \in \mathcal{K}.$$

It follows that

$$x = \sum \lambda_i v_i \qquad \text{where} \qquad \sum \lambda_i = 1.$$

Reorder the vertices $v_0, v_1, \dots, v_m$ so that

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_m.$$

In this case

$$\begin{aligned} x &= \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_m v_m \\ &= (\lambda_0 - \lambda_1) v_0 + 2(\lambda_1 - \lambda_2) \frac{v_0 + v_1}{2} + 3(\lambda_2 - \lambda_1) \frac{v_0 + v_1 + v_2}{3} + \dots \\ &= \mu_0 \hat{\sigma}_0 + \mu_1 \hat{\sigma}_1 + \mu_2 \hat{\sigma}_2 + \dots. \end{aligned}$$

Note that

$$\hat{\sigma}_p = \frac{v_0 + v_1 + \cdots + v_p}{p + 1}$$

is the Barycentre of

$$\sigma_p = \langle v_0, v_1, \dots, v_p \rangle \in \mathcal{K}.$$

On the other hand

$$\begin{aligned} \mu_0 + \mu_1 + \mu_2 + \dots &= (\lambda_0 - \lambda_1) + 2(\lambda_1 - \lambda_2) + 3(\lambda_2 - \lambda_3) + \dots \\ &= \lambda_0 + \lambda_1 + \lambda_2 + \dots \\ &= 1. \end{aligned}$$

Thus

$$x \in \Sigma = \langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_m \rangle \in \mathcal{L}. \qquad \square$$