## 10. Simplicial approximations

**Definition 10.1.** *Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes and let*

$$h\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

*be a continuous map.*

*We say that a simplicial map $f$ from $\mathcal{K}$ to $\mathcal{L}$ is a* ***simplicial approximation*** *to $h$ if*

$$h(\mathrm{St}(v)) \subset \mathrm{St}(f(v))$$

*for every vertex $v$ of $\mathcal{K}$.*

**Definition-Lemma 10.2.** *Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes and let*

$$h\colon |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

*be a continuous map.*

*We say that $h$ satisfies the* ***star condition*** *if for every vertex $v$ of $\mathcal{K}$ there is a vertex $w$ of $\mathcal{L}$ such that*

$$h(\mathrm{St}(v)) \subset \mathrm{St}(w).$$

*Then $h$ satisfies the star condition if and only if it has a simplicial approximation.*

*Proof.* One direction is clear. If $f$ is a simplicial approximation of $h$ then $h$ satisfies the star condition. Indeed, given $v$ if $w = f(v)$ then

$$\begin{aligned} h(\mathrm{St}(v)) &\subset f(\mathrm{St}(v)) \\ &= \mathrm{St}(w). \end{aligned}$$

Now suppose that $h$ satisfies the star condition. Given a vertex $v$ of $\mathcal{K}$ by assumption there is a vertex $w$ of $\mathcal{L}$ such that

$$h(\mathrm{St}(v)) \subset \mathrm{St}(w).$$

We define

$$f(v) = w.$$

This gives us a map on vertices.

Suppose that $\sigma$ is a simplex of $\mathcal{K}$. Pick $x$ in the interior of $\sigma$. Then $h(x)$ belongs to the interior of some simplex $\tau$.

Suppose that

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle.$$

Then

$$h(x) \in h(\mathrm{St}(v_i)) \subset \mathrm{St}(f(v_i))$$

Thus the barycentric coordinates of $h(x)$ are positive for each $f(v_i)$. But then $f(v_i)$ is vertex of $\tau$.

This gives us a map of the abstractions of $\mathcal{K}$ and $\mathcal{L}$ and in turn this means we can extend $f$ to a simplicial map. By construction $f$ is a simplicial approximation of $h$. □

**Definition-Lemma 10.3.** *Two simplicial maps $f$ and $g$ of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ are **contiguous** if for every simplex*

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle.$$

*of $\mathcal{K}$ the vertices $f(v_i)$ and $g(v_i)$ all belong to the same simplex $\tau$ of $\mathcal{L}$.*

*If $f$ and $g$ are simplicial approximations of the same continuous map then $f$ and $g$ are contiguous.*

*Proof.* We follow the proof of (10.2). If we pick $x$ in the interior of $\sigma$ then $h(x)$ belongs to the interior of some simplex $\tau$. It follows that $f(v_i)$ and $g(v_i)$ both belong to $\tau$. But then $f$ and $g$ are contiguous. □

**Theorem 10.4.** *Let $f$ and $g$ be two simplicial maps of two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$.*

*If $f$ and $g$ are contiguous then they are chain homotopic. In particular they induce the same map on homology.*

*Proof.* We want to define a group homomorphism

$$P_m \colon C_m(\mathcal{K}) \longrightarrow C_{m+1}(\mathcal{L})$$

such that

$$\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m = f_* - g_*.$$

It is enough to define $P_m$ for every simplex

$$\sigma = \langle v_0, v_1, \dots, v_m \rangle.$$

Let $\mathcal{L}(\sigma) \subset \mathcal{L}$ be the simplicial complex given by the simplex generated by

$$\{ f(v_i) \,|\, 0 \leq i \leq m \} \cup \{ g(v_i) \,|\, 0 \leq i \leq m \}.$$

and all its faces. We will construct $P_m$ such that

$$P_m(\sigma) \in C_m(\mathcal{L}(\sigma)).$$

We proceed by induction on $m$.

For $m = 0$, a 0-simplex is the same as a vertex $v$. By assumption $f(v)$ and $g(v)$ belong to the same simplex $\tau$. There are two cases. If $f(v) \neq g(v)$ then we define

$$P_0(v) = \langle g(v), f(v) \rangle.$$

In this case

$$\begin{aligned}(\partial_1 \circ P_0 + P_{-1} \circ \partial_0)(v) &= (\partial_1 \circ P_0)(v) + (P_{-1} \circ \partial_0)(v) \\ &= \partial_1 \langle g(v), f(v) \rangle \\ &= f(v) - g(v) \\ &= f_*(v) - g_*(v) \\ &= (f_* - g_*)(v).\end{aligned}$$

The other case is when $f(v) = g(v)$. In this case we simply put $P(v) = 0$. Then both sides of the equation above are zero, so we have equality in this case as well. This gives the case $m = 0$.

Now suppose that we have defined $P_k$ for all $k$-chains, where $k < m$. Suppose that $\sigma$ is an $m$-simplex

$$\langle v_0, v_1, \dots, v_m \rangle.$$

Consider

$$\gamma = f_*\sigma - g_*\sigma - P_{m-1}(\partial_m \sigma).$$

Note that $\gamma$ makes sense, as $\partial_m \sigma$ is an $(m-1)$-chain and we have already defined $P_{m-1}$. We check that $\gamma$ is a cycle:

$$\begin{aligned}\partial_m \gamma &= \partial_m f_* \sigma - \partial_m g_* \sigma - \partial_m (P_{m-1}(\partial_m \sigma) \\ &= f_*(\partial_m \sigma) - g_*(\partial_m \sigma) - (\partial_m \circ P_{m-1})(\partial_m \sigma) \\ &= f_*(\partial_m \sigma) - g_*(\partial_m \sigma) - f_*(\partial_m \sigma) - g_*(\partial_m \sigma) + P_{m-2}(\partial_{m-1})(\partial_m \sigma) \\ &= P_{m-2}((\partial_{m-1} \circ \partial_m)\sigma) \\ &= P_{m-2}(0) \\ &= 0.\end{aligned}$$

As the $m$th homology of $\mathcal{L}(\sigma)$ is zero,

$$H_m(\mathcal{L}(\sigma)) = 0,$$

it follows that $\gamma$ is a boundary, so that there is a $(m+1)$-chain $\beta \in C_{m+1}(\mathcal{L}(\sigma))$ such that $\partial_{m+1}\beta = \gamma$. We define $P_m(\sigma) = \beta$. With this choice we have

$$\begin{aligned}(\partial_{m+1} \circ P_m + P_{m-1} \circ \partial_m)(\sigma) &= (\partial_{m+1} \circ P_m)(\sigma) + (P_{m-1} \circ \partial_m)(\sigma) \\ &= \partial_{m+1}\beta + (P_{m-1} \circ \partial_m)(\sigma) \\ &= \gamma + P_{m-1}(\partial_m \sigma) \\ &= f_*\sigma - g_*\sigma - P_{m-1}(\partial_m \sigma) + P_{m-1}(\partial_m \sigma) \\ &= f_*(\sigma) - g_*(\sigma) \\ &= (f_* - g_*)(\sigma).\end{aligned}$$

This completes the induction and the construction of $P_m$. □

Note that if a continuous map $h$ between two triangulable topological spaces has a simplicial approximation then we can define a pushforward on homology. Indeed pick one simplicial approximation and use the fact that the pushforward of homology does not depend on the choice of simplicial approximation.