1. Introduction

Algebraic topology provides discrete invariants to distinguish between different topological spaces. In this introduction we try to motivate some of the definitions and ideas in this course.

I will talk about motivation from three seemingly disparate areas.

The first comes from calculus. Consider the 1-form

$$\omega = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}$$

on the punctured plane $\mathbb{R}^2 \setminus \{ 0 \}$. Note that we can differentiate forms formally. If

$$\omega = u\,\mathrm{d}x + v\,\mathrm{d}y$$

$$\begin{aligned} \mathrm{d}\omega &= u_y \mathrm{d}y \wedge \mathrm{d}x + v_x\,\mathrm{d}x \wedge \mathrm{d}y \\ &= -u_y \mathrm{d}x \wedge \mathrm{d}y + v_x\,\mathrm{d}x \wedge \mathrm{d}y \\ &= (v_x - u_y)\mathrm{d}x \wedge \mathrm{d}y, \end{aligned}$$

a 2-form.

In our case ω is closed, meaning that

$$\mathrm{d}\omega = 0.$$

This translates to the condition that

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) &= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) &= \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

are equal. But ω is not exact, meaning that there is no function f such that

$$\begin{aligned} \omega &= \mathrm{d}f \\ &= f_x \mathrm{d}x + f_y \mathrm{d}y. \end{aligned}$$

In other words we cannot solve the partial differential equation

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}.$$

Perhaps the easiest way to check this is to integrate ω around the unit circle. The denominator is 1 and if we use the parameterisation where t represents radians then

$$\begin{aligned}
\omega &= -\sin t \,\mathrm{d}\cos t + \cos t \,\mathrm{d}\sin t \\
&= \sin^2 t \,\mathrm{d}t + \cos^2 t \,\mathrm{d}t \\
&= (\sin^2 t + \cos^2 t)\,\mathrm{d}t \\
&= \mathrm{d}t.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{S^1} \omega &= \int_0^{2\pi} \mathrm{d}t \\
&= 2\pi.
\end{aligned}$$

If ω were exact then it is easy to see that the line integral around a closed path should be zero.

Note that every exact form is closed. Indeed this just reduces to the fact that the mixed partials

$$f_{xy} = f_{yx}$$

are equal.

Of course these observations are key to the study of differential equations. But one can also view this from the point of view of topology. The existence of closed forms that are not exact points to the existence of holes (in this case the origin).

The second motivation comes from graph theory. Euler proved a famous result that connects the number of vertices edges, and faces of a planar graph:

Theorem 1.1 (Euler's Formula)**.** *Let G be a connected planar graph with v vertices, e edges and f faces.*

Then

$$v - e + f = 2.$$

Proof. Suppose that G contains a cycle. Let ab be an edge of the cycle and let $G' = G - ab$ be the graph you get by deleting the edge ab. Suppose that the number of vertices, edges and faces of G' is v', e' and f'.

Since we deleted an edge of a cycle, G' is still connected. Note that the edge e is the boundary of two faces. So G' has the same number of vertices as G, one less edge and one less face

$$v' = v, \qquad e' = e - 1 \qquad \text{and} \qquad f' = f - 1.$$

It follows that

$$\begin{aligned} v' - e' + f' &= v - (e - 1) + (f - 1) \\ &= v - e + f. \end{aligned}$$

By induction, we may therefore assume that G is a tree (that is, G is a connected graph with no cycles). In this case $f = 1$.

Note that a tree always has at least one vertex a of degree 1. Let $G' = G - a$ be the graph you get by deleting a. Then G' is connected, so that it is a tree. G' has one less vertex and one less edge than G,

$$v' = v - 1, \qquad e' = e - 1 \qquad \text{and} \qquad f' = f = 1.$$

Thus

$$\begin{aligned} v' - e' + f' &= (v - 1) - (e - 1) + f \\ &= v - e + f. \end{aligned}$$

By induction, we are therefore reduced to the case when G has one vertex. In this case

$$v = 1, \qquad e = 0 \qquad \text{and} \qquad f = 1$$

so that

$$\begin{aligned} v - e + f &= 1 - 0 + 1 \\ &= 2. \end{aligned}$$

□

One can view Euler's formula as saying something about planar graphs. More interestingly for us we can view Euler's formula as saying something about S^2. Note also the importance of the alternating sum.

The final motivation comes from the last question on the final exam of Math 190A. The question was whether

$$[0, 1) \times (0, 1) \qquad \text{and} \qquad [0, 1) \times [0, 1]$$

are homeomorphic.

They are in fact homeomorphic. One way to exhibit a homeomorphism is to replace the square by the unit disc by the usual sort of argument and then use polar coordinates to move the boundary around.

Here is another way. Let's take the closure of the unit square. In the first instance we delete three sides and in the second one. We can use colour coding to keep track of deleted vs undeleted edges. Let red denote a deleted side and blue a side that has not been deleted.

Note that if we delete a side then we also delete the two corresponding vertices. Thus the union of the red edges are homeomorphic to $[0, 1]$ and the union of the blue edges are homeomorphic to $(0, 1)$.

Let the difference between the number of red sides minus the number of blue sides equal the warmth. We start with two squares, one of warmth two and one of warmth minus two. Let's suppose we pick a diagonal and divide our squares into two triangles. Note that the hypotenuses of the new triangles is the diagonal. One side is coloured red and the other is coloured blue.

We can always assign colours so that the square of warmth two becomes two triangles of warmth one and vice-versa for the square of warmth minus two, that is, we can arrange to end with two triangles of warmth minus one.

Now note that we can divide a triangle of warmth one into two triangles, one of warmth one and one of warmth minus one. The trick is to pick a point of the blue side and cut from this point to the other vertex. If we play the same trick with a triangle of warmth minus one then we are down to showing that any triangles of the same warmth are homeomorphic, by a homeomorphism which sends red points to red points, blue points to blue points and which patches across the sides.

This involves a fun piece of affine geometry. An **affine linear transformation** is a linear map followed by a translation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \text{given by} \qquad x \longrightarrow Ax + b,$$

where A is a real $m \times n$ matrix and $b \in \mathbb{R}^m$. It is easy to check that the composition of affine linear transformations is an affine linear transformation and that an affine linear transformation is invertible if and only if A is invertible.

The set of all invertible linear transformations is a group G under composition (as functions). G acts on $\mathbb{R}^2$ in the obvious way. We say three points p_1, p_2 and p_3 are in affine linear general position if they are not collinear. Let S be the set of all sequences of three points in linear general position.

The key result is that G acts precisely transitively on S, which is to say that given two elements of S there is exactly one affine linear transformation taking one sequence of points to the other.

It is easy to check that G does act on S. The translation of three collinear points is collinear and similarly for an invertible linear transformation. Thus collinearity (and non-collinearity) are preserved by G.

Using just a little bit of group theory, note that to show G acts precisely transitively on S, it suffices to show that everything is in the orbit of $(0,0)$, $(1,0)$ and $(0,1)$ and that the stabiliser of this point is trivial.

Note that the transformation

$$x \longrightarrow x + q_1$$

takes the origin to q_1. Thus there is no harm in assuming that $q_1 = 0$. In this case the condition that the points $(0,0)$, q_2 and q_3 are not collinear becomes the condition that the vectors q_2 and q_3 are independent. But there is a unique linear map that sends one basis to another.

In fact one can prove a little bit more. If we restrict an affine linear transformation to an affine linear subspace then we get an affine linear transformation (since restriction is the same as composing with the natural inclusion which is affine linear). If we restrict to a line then the natural map is determined by its action on two distinct points.

Hence the affine linear map that carries p_1 to q_1, p_2 to q_2 and p_3 to q_3, when restricted to the line segment connecting p_1 to p_2 does not depend on p_3 and q_3.

It follows that we can match up triangles and sides in a compatible way, so that we can patch everything together to get a homeomorphism.

Let me end the introduction by taking a completely different much more abstract perspective on algebraic topology. There are going to be three types of invariants one can write down which we will see in this course. One can view all three as functors from a subcategory of the category of topological spaces to the some algebraic category.

Roughly speaking a functor from one category to another sends objects to objects and morphisms to morphisms.

The first thing we will look at is homology. The first collection of topological spaces we will look at are simplicial complexes. These are the natural generalisations of triangles and triangulations. There is one homology group $H_n(X)$ for each natural number $n \geq 0$, which are either vector spaces or abelian groups (depending on the coefficients).

The second is the fundamental group. To any topological space one looks at the collection of all loops (based at one point) and from this one can construct a group (not necessarily abelian) $\pi_1(X)$.

The third is cohomology (let's say of a CW-complex; there are many different cohomology theories). To any CW-complex (take simplices and glue them together via more complicated maps than just affine linear transformations) one associates a (commutative) ring (so more algebraic structure),

$$\bigoplus_{n \in \mathbb{N}} H^n(X).$$

Cohomology is a contravariant functor (the arrows go the other way).

All three theories are quite closely related. Homology and cohomology are basically dual theories. The fundamental group carries a lot of structure but then it is very hard to compute. Computing homology and cohomology are relatively easy and there are lots of interesting invariants one can construct from these (for example, the Euler characteristic).