Oct 3, 2025 GE and LU factorization

Elementary transformations (do not change solution)

① Add a multiple of one row to another row

$$R_i \to R_i + cR_j$$

② Switch rows

$$R_i \leftrightarrow R_j \quad \text{(related to pivoting)}$$

③ Multiply a row by a non-zero scalar

$$R_i \to cR_i$$

Augmented matrix $[A \mid b]$

Ex. $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$

Step 0 $[A^{(1)} \mid b^{(1)}] = [A \mid b]$

Step 1 Make desired zeros in the first column

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ \boxed{2} & 1 & -3 & 5 \\ \boxed{4} & -7 & 1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & \boxed{1} & -3 & -1 \end{array}\right]$$

$$= [A^{(2)} \mid b^{(2)}]$$

Step 2. Make desired zeros in the second column

$$[A^{(2)} | b^{(2)}] \xrightarrow{R_3 \to R_3 - \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$= [A^{(3)} | b^{(3)}]$$

Summary.

① If $A \in \mathbb{R}^{n\times n}$, start from $A^{(1)} = A$, $b^{(1)} = b$,

We get an upper triangular matrix $[A^{(n)} | b^{(n)}]$ in $n-1$

elimination steps if $\underline{a_{kk}^{(k)} \neq 0}$, $k = 1, \cdots, n-1$ Note: $a_{kk}^{(k)}$ is called a "pivot"

$$A^{(k)} = \begin{pmatrix} \ddots & & \\ & a_{kk}^{(k)} & \\ & & \ddots \end{pmatrix}$$

② To get $[A^{(k+1)} | b^{(k+1)}]$ from $[A^{(k)} | b^{(k)}]$

$$= \left[\begin{array}{cccccc|c} * & & - & - & - & * & * \\ & \ddots & & & & \vdots & \vdots \\ & & * & & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{kk}^{(k)} & & \vdots & \vdots \\ \vdots & & \vdots & \boxed{*} & \ddots & \vdots & \vdots \\ 0 & & 0 & \boxed{*} & & * & * \end{array}\right]$$

$$R_i \to R_i - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} R_k \qquad \text{for } i > k$$

i.e., $$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)} \quad \text{for } i > k,\ j \geq k$$

$$b_i^{(k+1)} = b_i^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} b_k^{(k)}$$

$\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$ is called a multiplier, we denote it by l_{ik}

Pseudo-code GE (without pivoting)

```
for k = 1 : n-1
    for i = k+1 : n
        l_ik = a_ik / a_kk ;  (if a_kk = 0, GE fails)
        a_ik = 0 ;
        for j = k+1 : n
            a_ij = a_ij - l_ik * a_kj
        end
        b_i = b_i - l_ik * b_k ;
    end
end
```

FLOPS ?

Answer

$\frac{2}{3}n^3 + O(n^2)$

(Discussion)

Ex. Let $L_1 = \begin{bmatrix} 1 & & & & 0 \\ -l_{21} & 1 & & & \\ -l_{31} & 0 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ -l_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$

$$L_1 \cdot A = \begin{bmatrix} 1 & & & & 0 \\ -l_{21} & 1 & & & \\ -l_{31} & 0 & \ddots & & \\ \vdots & \vdots & & 1 & \\ & & & & \ddots \\ -l_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} - l_{21}a_{11} & a_{22} - l_{21}a_{12} & \cdots & a_{2n} - l_{21}a_{1n} \\ a_{31} - l_{31}a_{11} & \cdots & - & \\ \vdots & & & \\ a_{n1} - l_{n1}a_{11} & \cdots & - & - \end{bmatrix}$$

$$\left(R_i \rightarrow R_i - l_{i1} R_1 , \text{ for } i > 1 \right)$$

In general, all GE steps can be represented by

$$L_{n-1} \cdots L_2 L_1 A = U$$

<u>Fact</u> $$A = (L_{n-1} \cdots L_2 L_1)^{-1} U$$

$$= (L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1}) U$$

$$= LU$$

where $$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & \ddots & \\ \vdots & & & \ddots \\ l_{n1} & l_{n2} & \cdots \; l_{n,n-1} & 1 \end{bmatrix}$$

Pseudocode LU factorization

Input : A . Output : lower triangular L and upper triangular U

```
L = I;
for k = 1 : n-1
    for i = k+1 : n
        L(i,k) = A(i,k) / A(k,k);
        A(i,k) = 0;
        for j = k+1 : n
            A(i,j) = A(i,j) - L(i,k) * A(k,j);
        end
        b(i) = b(i) - L(i,k) * b(k);
    end
end
U = A;
```