Oct 1, 2025

Last time.

**Def.** <u>Column space</u> : All combinations of the columns of $A$, denoted by $C(A)$

$\dim(C(A)) = \mathrm{rank}(A)$

**Def.** <u>Null space</u> : All $x$ such that $Ax = 0$, denoted by $N(A)$.

$A = [a_1 \ a_2 \ a_3] \in \mathbb{R}^{3\times 3}$

$a_1, a_2, a_3$ linearly independent $\iff C(A) = \mathbb{R}^3$ $(\mathrm{rank}(A) = 3)$

$\iff Ax = b$ has a solution $x$ for every $b \in \mathbb{R}^3$

In fact, $x$ is unique. why?

Rank-Nullity theorem

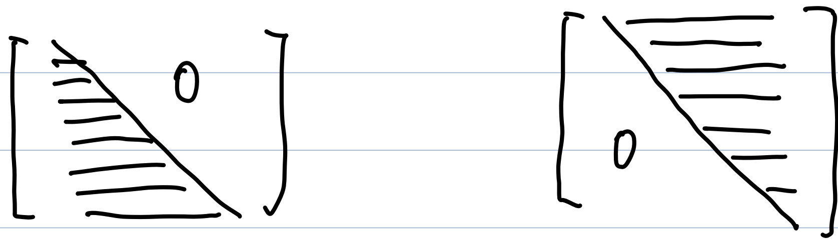
$$\boxed{\mathrm{rank}(A) + \mathrm{nullity}(A) = \#\text{columns of } A}$$

$\Rightarrow N(A) = \{0\} \Rightarrow Ax = b$ has only one solution for a given $b \in \mathbb{R}^3$

Remark. These are also equivalent to ① $A$ is invertible

② $\det(A) \neq 0$

Triangular matrices



lower trianglular, $L$ 　　　　 upper triangular, $U$

ex. $\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ 　　　　 $\begin{bmatrix} 2 & 4 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

unit lower triangular 　　　　 unit upper triangular

ex. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ 　　　　 $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

properties.

1. If $L_1$ and $L_2$ are (unit) lower triangular, $L_1 L_2$ is (unit) lower triangular

   If $U_1$ and $U_2$ ·· (unit) upper ··· , $U_1 U_2$ is (unit) upper ···

2. $$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

   $$\det(L) = l_{11} \cdot l_{22} \cdot l_{33} \cdots l_{nn} = \prod_{i=1}^{n} l_{ii}$$

   $$\det(L) \neq 0 \Leftrightarrow l_{ii} \neq 0, \ i = 1, \cdots, n$$

3. $L^{-1}$ is lower triangular, $U^{-1}$ is upper triangular.

Why?

- preparation for Gaussian Elimination (GE)

- easy to solve.

Solve $Lx = b$, assume $l_{ii} \neq 0, i=1,\cdots,n$

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$l_{11} \cdot x_1 = b_1 \Rightarrow x_1 = b_1 / l_{11}$$

$$l_{21} \cdot x_1 + l_{22} \cdot x_2 = b_2 \Rightarrow x_2 = \frac{b_2 - l_{21} \cdot x_1}{l_{22}}$$

$$l_{31} \cdot x_1 + l_{32} \cdot x_2 + l_{33} \cdot x_3 = b_3 \Rightarrow x_3 = \frac{b_3 - l_{31} \cdot x_1 - l_{32} \cdot x_2}{l_{33}}$$

$$\vdots$$

Start from top to go to bottom, and for equation $i$ ($i=1,\cdots,n$)

let $x_i = \dfrac{b_i - \sum_{j=1}^{i-1} l_{ij} \cdot x_j}{l_{ii}}$ [Note: $i=1$, $\sum_{j=1}^{0} = 0$. empty]

This is called Forward Substitution

Pseudo-code ($Lx = b$)

```
for i = 1:n
    x(i) = b(i);
    for j = 1:i-1
        x(i) = x(i) - L(i,j) * x(j)    } 2(i-1) FLOPS
```

```
        end
    If L(i,i) = 0
        error (matrix not invertible);
    else
        x(i) = x(i) / L(i,i);          1 FLOP
    end
end
```

For each $i$, $2(i-1)+1 = 2i-1$ FLOPS

Total FLOPS $= \sum_{i=1}^{n} (2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$

$$= 2\cdot\frac{n(n+1)}{2} - n$$

$$= n(n+1) - n$$

$$= \underline{n^2}$$

Question: FLOP count of diagonal system?

$$\rightarrow \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad n \text{ FLOPS}$$

Exercise: Backward substitution for $Ux = b$

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Gaussian Elimination · Intro

We want to solve $Ax=b$, $A$ is $n\times n$ and invertible.

Strategy: Tranform $Ax=b$ to an equivalent system (meaning: same solution)
$Ux=y$ with $U$ upper triangular,
then do backward substitution.

Elementary transformations (do not change solution)

① Add a multiple of one row to another row

$$R_i \to R_i + cR_j$$

② Switch rows

$$R_i \leftrightarrow R_j \quad \text{(related to pivoting)}$$

③ Multiply a row by a non-zero scalar

$$R_i \to cR_i$$

Augmented matrix $[A \mid b]$

Ex. $A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix}$, $b=\begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$

Step 0 $[A^{(1)} \mid b^{(1)}] = [A \mid b]$

Step 1 Make desired zeros in the first column

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$