MATH 140A FOUNDATIONS OF REAL ANALYSIS
FALL 2024
PRACTICE PROBLEMS FOR MIDTERM 2

1. Suppose that $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. Show that $|x| < \varepsilon$ if and only if $-\varepsilon < x < \varepsilon$.
2. (a) Show that every convergent sequence is bounded.
   (b) Is every bounded sequence convergent? If so, prove it. If not, provide a counterexample with justification.
3. Suppose that (x_n) is a sequence of positive numbers that converge to a real number $x > 0$. Prove that $\sqrt{x_n} \to \sqrt{x}$.
4. Set $x_1 = 3$ and $x_{n+1} = \frac{8x_n}{2x_n+9}$ for $n \in \mathbb{N}$.
   Find $\lim_{n\to\infty} x_n$.
5. Let (X, d) be a metric space and let $E, F \subset X$. Assume that E is connected, F is connected, and $E \cap F \neq \emptyset$. Prove that $E \cup F$ is connected.
6. Let (X, d) be a metric space, let $K \subset X$ be compact, and let $F \subset X$ be closed. Prove that if $K \cap F = \emptyset$ then there is $r > 0$ so that for all $p \in K$, we have $N_r(p) \cap F = \emptyset$.
7. Let (X, d) be a metric space. Show that a Cauchy sequence is bounded.
8. Let (X, d) be a metric space, $a \in X$, $K \subset X$ such that K is compact and $a \notin K$. Show that there there exists a neighborhood V of a and an open set $W \supset K$ such that $V \cap W \neq \emptyset$.
9. If $a_n \neq 0$ and $L := \lim |\frac{a_{n+1}}{a_n}|$ exists and satisfies $L < 1$, then $\lim_n a_n = 0$.
   Hint: select $l \in (L, 1)$ and obtain N so that $|a_{N+1}| < l|a_n|$ for $n \geq N$. Then $|a_n| < l^{n-N}|a_N|$ for $n > N$.
10. Prove $\lim \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.