MATH 140A FOUNDATIONS OF REAL ANALYSIS
FALL 2024
PRACTICE PROBLEMS FOR THE FINAL

1. Let $\mathbf{x}_n = (x_{1,n}, x_{2,n}, \ldots, x_{d,n})$. Show that a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$ converges to $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ if and only if each component $x_{i,n}$ converges to x_i in $\mathbb{R}$.
2. Show that the composition of continuous functions is continuous using an ϵ-δ argument.
3. (a) Prove that f is continuous if and only if the preimage of each open set is open.
   (b) Show that $f^{-1}(F^c) = (f^{-1}(F))^c$ for any set F.
   (c) Show that f is continuous if and only if the preimage of each closed set is closed.
4. Suppose $(x_n) \subset E$ is a sequence in a metric space. Show that if every subsequence x_{n_k} has a convergent subsequence $x_{n_{k_j}}$ that converges to ℓ, then $x_n \to \ell$.
5. Show that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
6. Show that the set of subsequential limits of $[0,1] \cap \mathbb{Q}$ is all of $[0,1]$.
7. Use the intermediate value theorem to show that m^{th} roots exists. That is, let $y > 0$ and $m \in \mathbb{N}$ with $m \geq 2$. Show there exists $x > 0$ so that $x^m = y$.
8. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be an injective and continuous function. Prove that f is strictly monotone.
9. Let (X, d) be a compact metric space and let $f : X \to X$ be a function with the following property. There exists some $0 < c < 1$ so that
$$d(f(x), f(y)) \leq c\, d(x, y) \quad \text{for all } x, y \in X.$$
   (a) Prove that f is continuous.
   (b) Let $x \in X$ and consider the sequence
$$f(x), f(f(x)), \ldots$$
   Prove that this sequence is Cauchy.
   (c) Use part (b) to conclude that there exists a unique $x_0 \in X$ so that $f(x_0) = x_0$.
   (d) Prove part (c) using a theorem nested sequence of compact sets by considering $X \supset f(X) \supset f(f(X)) \supset \cdots$.
10. (a) Suppose f is a continuous function on $\mathbb{R}$ and assume that
$$\lim_{x \to \infty} f(x) = \infty = \lim_{x \to -\infty} f(x).$$
   Prove that there exists some $y \in \mathbb{R}$ such that $f(y) \leq f(x)$ for all $x \in \mathbb{R}$.

(b) Let n be an even natural number and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a polynomial of degree n. Prove that f has a minimum, i.e. there exists some $y \in \mathbb{R}$ such that $f(y) \leq f(x)$ for all $x \in \mathbb{R}$.

11. Let $p \in [0,1] \times [0,1]$, and let $C = [0,1] \times [0,1] \setminus \{p\}$.

    (a) Let $x, y \in C$. Show that there exists a continuous function $f : [0,1] \to C$ so that $f(0) = x$ and $f(1) = y$.

    (b) Use part (a) and an argument similar to Rudin problem 21, page 44, to show that C is connected.

12. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbb{R}$ into $\mathbb{R}$ is monotonic.
13. Suppose that α is irrational. Show that $(\{n\alpha\})_{n\in\mathbb{N}}$ is dense in $[0,1)$ where $\{x\} := x - \lfloor x \rfloor$.