Homework 9 due Thursday, December 5 by 11:59 pm Pacific Time.

Rudin, Chapter 4 (page 98), problems # 18, 20, 21 (only the first part), 22.

A. Let $f : [0,1] \to \mathbb{R}$ be a function and let $a \in [0,1]$. Suppose there is some $\ell \in \mathbb{R}$ so that

$$\lim_{x \to a} f(x) = \ell.$$

Define

$$g(x) = \begin{cases} f(x) & x \neq a \\ \ell & x = a \end{cases}$$

Prove that g is continuous at a.

B. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function so that $\lim_{x\to\infty} f(x) = \ell_1$ and $\lim_{x\to-\infty} f(x) = \ell_2$, where $\ell_1, \ell_2 \in \mathbb{R}$. Prove that f is uniformly continuous.

C. Let (X, d) be a metric space and let $f : X \to X$ be a function. Suppose there exist $C > 0$ and $\alpha > 0$ so that

$$d(f(x), f(y)) \leq C d(x,y)^\alpha \quad \text{for all } x, y \in X.$$

Prove that f is uniformly continuous.

D. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bijection, prove that f is a homeomorphism.

(Hint: Use the fact that connected subsets of $\mathbb{R}$ are intervals to determine what $f([a,b])$ can be for a closed interval $[a,b]$.)

E. Let p be a prime number. For any integer $m \in \mathbb{Z}$ define

$$\nu_p(m) = \begin{cases} \text{power of } p \text{ in the prime factorization of } m & \text{if } m \neq 0 \\ \infty & \text{if } m = 0 \end{cases}$$

and define the following **norm** on $\mathbb{Q}$

$$|\frac{m}{n}|_p = \begin{cases} p^{\nu_p(n)-\nu_p(m)} & \text{if } \frac{m}{n} \neq 0 \\ 0 & \text{if } \frac{m}{n} = 0 \end{cases}.$$

(1) Compute $|\frac{75}{73}|_5$.
(2) Show that this definition is well defined, i.e. if $\frac{m}{n} = \frac{m'}{n'}$, then $|\frac{m}{n}|_p = |\frac{m'}{n'}|_p$.
(3) For any two $r, s \in \mathbb{Q}$ define $d(r,s) = |r - s|_p$. Show that $(\mathbb{Q}, d)$ is a metric space.
(4) Show that $\mathbb{Z}$ is a bounded subset of $\mathbb{Q}$ with respect to this metric.
(5) Does $\{p^n : n \in \mathbb{N}\}$ have a limit point in $\mathbb{Q}$ with respect to the distance d from part(c)?

The following problems are for your practice, and will not be graded.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}$$

(a) Prove that there exists some $\lambda \in \mathbb{R}$ such that $f(r) = \lambda r$ for all $r \in \mathbb{Q}$.
(b) Prove that if f is continuous at 0, then it is continuous at every $x \in \mathbb{R}$.
(c) Prove that if f is continuous at 0, then $f(x) = \lambda x$ for all $x \in \mathbb{R}$, where λ is the number given in (a).

(2) Let $f : [0, 1] \to [0, 1]$ be a function which satisfies

$$\lim_{y \to x} f(y) \text{ exists for all } x \in [0, 1].$$

Define $g(x) = \lim_{y \to x} f(y)$. Prove that g is continuous on $[0, 1]$.