Homework 7 due Friday, November 22, by 11:59 pm Pacific Time.

Rudin, Chapter 3 (page 78), problems # 6, 7, 8, 9, 10, 11(abc), 22, 23.

A. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. For every $N \in \mathbb{N}$ define

$$s_N = \sup\{x_n : n \geq N\} \quad \text{and} \quad r_N = \inf\{x_n : n \geq N\}.$$

Prove that

(1) $\{s_N\}$ and $\{r_N\}$ are bounded.
(2) $\{s_N\}$ is non-increasing, i.e., $s_1 \geq s_2 \geq \cdots$, and $\{r_N\}$ is nondecreasing, i.e., $r_1 \leq r_2 \leq \cdots$.
(3) Let $s = \lim s_N = \inf\{s_N : N \in \mathbb{N}\}$, and $r = \lim r_N = \sup\{r_N : N \in \mathbb{N}\}$. Prove that $s = \limsup x_n$ and $r = \liminf x_n$.

B. Evaluate the following limits

(1) $\lim_{n\to\infty} \sqrt[n]{n!}$.
(2) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$. (You may use, without a proof: $\lim_{n\to\infty}(\frac{n+1}{n})^n = e$.)

(Hint: Theorem 3.37 maybe helpful.)

C. Let (X, d) be a compact metric space and let $f : X \to X$ be a function with the following property. There exists some $0 < c < 1$ so that

$$d(f(x), f(y)) \leq c\, d(x, y) \quad \text{for all } x, y \in X.$$

Let $x \in X$ and consider the sequence

$$f(x), f(f(x)), f(f(f(x)))\ldots$$

Prove that this sequence is Cauchy.

The following problems are for your practice, and will not be graded.

(1) Rudin Chapter 3 (page 78), problems # 24, 25.

Interested students should look at 24 for an alternative way of constructing the real numbers without Dedekind cuts.

(2) Let $\sum b_n$ be a convergent series of real numbers, and let $\{a_n\}$ be a sequence of real numbers which is bounded below. Assume further that

$$a_{n+1} \leq a_n + b_n.$$

Prove that $\{a_n\}$ converges.