Homework 6 due Friday, November 8, by 11:59 pm Pacific Time.

Rudin, Chapter 3 (page 78), problems # 1, 2, 3, 4, 5, 14(ab), 16(a), 20, 21.

A. Using the definition of the limit of a sequence prove the following ($\mathbb{R}$ is considered with the standard metric).

(1) $\lim_{n\to\infty} \frac{2n+1}{3n+5} = \frac{2}{3}$.

(2) $\lim\left(\frac{n}{n+1}\right)^2 = 1$.

B. Let (X, d) be a metric space, and let $\{p_n\}$ be a sequence in X. Assume that all the subsequences $\{p_{n_i}\}$ where $\mathbb{N} \setminus \{n_i : i\}$ is infinite converge (e.g. we are including sequences like $p_1, p_3, p_5, \ldots$ but not something like $p_2, p_3, p_4, p_5, \ldots$).

Prove that $\{p_n\}$ converges.

(Note that we are not assuming all the subsequences converge to the same limit, this indeed is the first thing you need to show.)

C. Let $\{a_n\}$ be a sequence of bounded real numbers. Prove that $\ell \in \mathbb{R}$ is a subsequential limit of $\{a_n\}$ if and only if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |a_n - \ell| < \varepsilon\}$ is infinite.

The following problems are for your practice, and will not be graded.

(1) (a) Prove that $\lim_{n\to\infty} \frac{4^n}{n!} = 0$.
(b) Let $a \in \mathbb{R}$, prove that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.