Homework 4. Due Friday, October 25, by 11:59 pm Pacific Time.

Rudin, Chapter 2 (page 43-45), problems # 9, 12, 14, 22, 23, 24, 29.

A. Let (X, d) be a metric space. For any two nonempty subsets $A, B \subset X$ define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Note that if $A \cap B \neq \emptyset$, then $d(A, B) = 0$. Prove or provide a counter example to the following statements.

(1) If A and B are two disjoint subsets of X, then $d(A, B) > 0$.
(2) If A and B are two separated subsets of X, then $d(A, B) > 0$. (Two subsets are called separated if $A \cap \bar{B} = \emptyset$ and $B \cap \bar{A} = \emptyset$)
(3) If A and B are two disjoint open subsets of X, then $d(A, B) > 0$.
(4) If A and B are two disjoint closed subsets of X, then $d(A, B) > 0$.
(5) If A and B are two disjoint compact subsets of X, then $d(A, B) > 0$.

B.

(a) Let (X, d) be a metric space. Prove that the closed neighborhood

$$\overline{N}_r(x) = \{y \in X : d(x, y) \leq r\}$$

is a closed set.
(b) Let $(\mathbb{R}^n, d)$ with the standard metric. Prove that the closure of the open neighborhood $N_r(x)$ is the closed neighborhood.
(c) Is it true in general that the closure of the open neighborhood $N_r(x)$ is the closed neighborhood? If true, prove it and if false, give a counter example?

The following problems are for your practice, and will not be graded.

(1) Let (X, d) be a metric space and let $K \subset X$ be a compact set. Let $x \in K^c$. Define $d(x, K)$ as in the previous problem, i.e.,

$$d(x, K) = \inf\{d(x, y) : y \in K\}.$$

Prove: there exists some $z \in K$ so that $d(x, K) = d(x, z)$.

(2) Show that every closed subset of $\mathbb{R}^n$ is an intersection of countably many open sets.

(3) Let $\{x_1, x_2, \ldots\}$ be a countable subset of $\mathbb{R}$. Assume that there are finitely many point $y_1, \ldots, y_m \in \mathbb{R}$ so that the following holds.

$$\forall \epsilon > 0,\ \exists N \text{ such that } \forall n > N,\ \exists 1 \leq \ell \leq m,\ |x_n - y_\ell| < \epsilon.$$

Set $K = \{x_1, x_2, \ldots\} \cup \{y_1, \ldots, y_m\}$.
(a) Prove that K is compact.

(b) Using the definition of a compact set, prove that K is compact. (In part (b) you are supposed to use the definition and not the Heine-Borel theorem.)

(4) Let (X_1, d_1) and (X_2, d_2) be two compact metric spaces. Let $X = X_1 \times X_2$ and define

$$d\big((x_1, x_2), (y_1, y_2)\big) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Prove that (X, d) is a compact metric space.