HOMEWORK 3. DUE FRIDAY, OCTOBER 18, BY 11:59 PM PACIFIC TIME.

Rudin, Chapter 2 (page 43), problems 2, 3, 5, 6, 7, 8.

A. Prove that the set of all injections from the set of natural numbers to itself is uncountable.

B. Determine all the limit points of the following sets in $\mathbb{R}$ (with the standard metric) and determine whether the sets are open or closed (or neither).

(1) All integers.
(2) The interval $(a, b]$.
(3) All rational numbers.
(4) All numbers of the form $(-1)^n + \frac{1}{m}$, where $m, n = 1, 2, \ldots$
(5) All numbers of the form $\frac{1}{n} + \frac{1}{m}$ where $m, n = 1, 2, \ldots$
(6) All numbers of the form $\frac{(-1)^n}{1+(1/n)}$, where $n = 1, 2, \ldots$

C. In this problem you may use without a proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ with the standard metric.

(1) Show that $A = \{a + b\mathbf{i} : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$ with the usual metric on $\mathbb{C}$.
(2) Show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ with the usual metric on $\mathbb{R}^n$.
(3) Let A be as in part (a). Show that A^n is dense in $\mathbb{C}^n$ with the usual metric on $\mathbb{C}^n$.

Extra practice problems: Extra Problems are for your practice; please do not hand them in. However, it is extremely important that you feel comfortable with these problems as some of them may appear on the exams.

Let
$$\ell^\infty = \Big\{(a_1, a_2, \ldots) : a_n \in \mathbb{R} \text{ and } \sup\{|a_n| : n \in \mathbb{N}\} < \infty\Big\}.$$
We use componentwise addition and scalar multiplication on ℓ^∞. For any $(a_n) \in \ell^\infty$, define
$$\big\|(a_n)\big\|_\infty = \sup\{|a_n| : n \in \mathbb{N}\}.$$
Define
$$\mathrm{d}\big((a_n), (b_n)\big) = \big\|(a_n - b_n)\big\|_\infty$$
for any two sequences (a_n) and (b_n).

(1) Show that ℓ^∞ is uncountable.
(2) Show that $\mathrm{d}\big((a_n), (b_n)\big)$ is a finite number for any two (a_n) and (b_n) in ℓ^∞.
(3) Show that $\big(\ell^\infty, \mathrm{d}\big)$ is a metric space. e.g., $\mathbb{R}$ with the usual metric is separable.)