Homework due Friday, October 11, at 11:59 pm Pacific Time.

Rudin, Chapter 1 (page 21), problems 8, 9, 10, 13, 14, 15, 17.

A. Let $a, b, c, d \in \mathbb{R}$ and assume $a < b$ and $c < d$. Give an explicit one-to-one correspondence between

(1) The points of the two open intervals (a, b) and (c, d).
(2) The points of the two closed intervals $[a, b]$ and $[c, d]$.
(3) The points of the closed interval $[a, b]$ and the open interval (c, d).
(4) The points of the closed interval $[a, b]$ and $\mathbb{R}$.

B. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let $X = X_1 \times X_2$.

(1) Define
$$\rho_\infty((x_1, x_2), (y_1, y_2)) = \max\{\mathrm{d}_1(x_1, y_1), \mathrm{d}_2(x_2, y_2)\}.$$
Prove that (X, ρ_∞) is a metric space.

(2) Define
$$\rho_1((x_1, x_2), (y_1, y_2)) = \mathrm{d}_1(x_1, y_1) + \mathrm{d}_2(x_2, y_2).$$
Prove that (X, ρ_1) is a metric space.

(3) Define
$$\rho_2((x_1, x_2), (y_1, y_2)) = \Big(\mathrm{d}_1(x_1, y_1)^2 + \mathrm{d}_2(x_2, y_2)^2\Big)^{1/2}.$$
Prove that (X, ρ_2) is a metric space.

(4) Let (X_i, d_i) be $\mathbb{R}$ with respect to the standard metric for $i = 1, 2$. Describe $N_1(0)$ in $X = \mathbb{R}^2$ with respect the metrics ρ_∞, ρ_1, and ρ_2.

Extra practice problems: Please do not hand in the problems below this line. However, it is extremely important that you feel comfortable with these problems as some of them may appear on the exams.

(C) Suppose that $x \in \mathbb{R}$ and $r \in \mathbb{R}$ with $r > 0$. Show that $|x| < r$ if and only if $-r < x < r$.

(D) For any positive integer n, define
$$\mathbb{C}^n = \{(z_1, \dots, z_n) : z_j \in \mathbb{C}\}.$$
For any two elements $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, define
$$\mathbf{z} + \mathbf{w} = (z_1 + w_1, \dots, z_n + w_n).$$

For any $\mathbf{z} = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$ and any $\lambda \in \mathbb{C}$, define

$$\lambda\mathbf{z} = (\lambda z_1, \ldots, \lambda z_n).$$

Define a scalar product on $\mathbb{C}^n$ as follows. Let $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be in $\mathbb{C}^n$, define

$$(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^{n} z_j \overline{w}_j.$$

Let $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ and let $\lambda \in \mathbb{C}$.

(1) Show that $(\lambda\mathbf{z}, \mathbf{w}) = \lambda(\mathbf{z}, \mathbf{w}) = (\mathbf{z}, \bar{\lambda}\mathbf{w})$.

(2) Show that $(\mathbf{z}, \mathbf{z})$ is a non-negative real number. Moreover, $(\mathbf{z}, \mathbf{z}) = 0$ if and only if $\mathbf{z} = (0, \ldots, 0)$.

(3) Show that $(\mathbf{z}, \mathbf{w}) = \overline{(\mathbf{w}, \mathbf{z})}$.

(4) Assume $\mathbf{w} \neq (0, \ldots, 0)$ and let $\mathbf{u} = \mathbf{z} - \frac{(\mathbf{z},\mathbf{w})}{(\mathbf{w},\mathbf{w})}\mathbf{w}$. Show that $(\mathbf{u}, \mathbf{w}) = 0$.

(5) Compute $(\mathbf{u}, \mathbf{u})$. Then use part (2) to prove the Cauchy-Schwarz inequality.