Math 114/214: Introduction to Computational Stochastics
Spring quarter 2025

Homework Assignment 3

Due: 3:00 pm, Monday, April 28, 2025

Instructions.

- Please submit your solution as a single PDF file to Gradescope.
- You are encouraged to type out your solution. If you write your solution, please write it neatly, and then scan it into a PDF file. Using cell phone cameras is discouraged due to low resolution.
- For each of the simulation problems, please include pseudocode or code of the simulation algorithm.

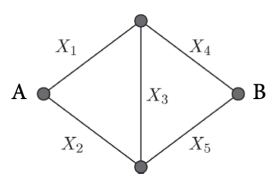
1. Let Z be a random variable with distribution F. Let $X_1, \ldots, X_N \sim F$ and $Y_1, \ldots, Y_N \sim F$ be such that $(X_1, Y_1), \ldots, (X_N, Y_N)$ are independent. Denote $A_N = (1/2N) \sum_{k=1}^{N} (X_k + Y_k)$. Show that

$$\mathbb{E}(A_N) = \mathbb{E}(Z) \qquad \text{and} \qquad \text{Var}\,(A_N) = \frac{1}{2N}\text{Var}\,(Z) + \frac{1}{2N}\text{Cov}\,(X_1, Y_1).$$

2. The figure below depicts a bridge network. All $X_1, \ldots, X_5$ are random length that are exponentially distributed with means $1, 1, 0.5, 2, 1.5$, respectively. (Recall that the mean of the exponential distribution Exp (λ) is $1/\lambda$.) The four allowable paths from A to B are $X_1 + X_4$, $X_2 + X_5$, $X_1 + X_3 + X_5$, and $X_2 + X_3 + X_4$. Estimate the expected length of the shortest path for the bridge network,

$$H(X_1, \ldots, X_5) = \min(X_1 + X_4, X_2 + X_5, X_1 + X_3 + X_5, X_2 + X_3 + X_4),$$

using both the Crude Monte Carlo estimator and the antithetic estimator, with the sample size of $N = 10,000$ and $N/2 = 5,000$, respectively. (See Example 2 of Lecture 10.)



3. Let $g(x) = \sqrt{x}e^{-x}$ $(x \in [0, 4])$. Estimate the integral $I = \int_0^4 g(x)\,dx$ with the following estimators:

   (1) A Crude Monte Carlo estimator: $I_N = (1/N) \sum_{i=1}^{N} 4g(X_i)$, where $X_1, \ldots, X_N \sim \mathcal{U}[0, 4]$ iid; (Note that the constant function $f(x) = 1/4$ is the probability density function for $\mathcal{U}[0, 4]$.)

   (2) An importance sampling estimator: $J_N = (1/N) \sum_{i=1}^{N} g(Y_i)/\varphi(Y_i)$, where $\varphi(x) = e^{-x}/(1 - e^{-4})$ is the importance function, and $Y_1, \ldots, Y_N \sim \varphi$ iid.

   Output the values of I_N and J_N, and the corresponding sample variance for each of these estimators, for $N = 100, 400, 900, 1600$. Discuss your simulation results.

4. Stochastic algorithms for approximating the maximum value of a continuous function f on $[0, 1]$. Fix a natural number N.

   (1) *Method A*. Generate $U_1, \ldots, U_N \sim \mathcal{U}[0, 1]$ iid. Set $M_A = \max\{f(U_i) : i = 1, \ldots, N\}$.

   (2) *Method B*. Pick up a natural number k so that N/k is an integer. Divide $[0, 1]$ into $[(i-1)/k, i/k]$ $(i = 1, \ldots, k)$. For each i, general N/k iid numbers uniformly distributed on $[(i-1)/k, i/k]$, and set m_i to be the maximum of f applied to these numbers. Finally, set $M_B = \max\{m_1, \ldots, m_k\}$.

   Both M_A and M_B approximate $\max_{x \in [0,1]} f(x)$. Find M_A and M_B for $f(x) = 1 - 4(x - 1/2)^2$ with $N = 1000$ and $k = 10$. Try some more N and k values to see if one method is better than the other.