**Math 114/214: Introduction to Computational Stochastics**
**Spring quarter 2025**

# Homework Assignment 2

**Due: 3:00 pm, Friday, April 18, 2025**

**Instructions.**

- Please submit your solution as a single PDF file to Gradescope.
- You are encouraged to type out your solution. If you write your solution, please write it neatly, and then scan it into a PDF file. Using cell phone cameras is discouraged due to low resolution.
- For each of the simulation problems, please include pseudocode or code of the simulation algorithm.

1. Let $X_1, \ldots, X_n$ be independent random variables in $\mathbb{R}$ with the same distribution. In particular, they have the same expectation $\mu$ and variance $\sigma^2$. Define the sample mean $\overline{X}_n$ and sample variance $S_n^2$ to be
$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \qquad \text{and} \qquad S_n^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X}_N)^2,$$
respectively. Show that the expectation $\mathbb{E}(S_n^2) = \sigma^2$.

2. Let $D = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ for some integer $d \geq 1$. Let $g : D \to \mathbb{R}$ be a countinuous function. Let $f : \mathbb{R}^d \to [0, \infty)$ be the PDF of a random variable $X \in \mathbb{R}^d$ such that $f = 0$ outside $D$. Suppose $X_1, \ldots, X_N \in D$ are independent random variables that are identically distributed according to $f$. Define
$$I = \int_D g(x) f(x)\, dx \qquad \text{and} \qquad I_N = \frac{1}{N}\sum_{k=1}^{N} g(X_k).$$
Note that $I = \mathbb{E}(g(X))$. Prove that
$$\mathrm{Var}\,(I_N) = \frac{1}{N}\mathrm{Var}\,(g(X)) = \frac{1}{N}\int_D \left[g(x) - I\right]^2 f(x)\, dx = \frac{1}{N}\left[\int_D (g(x))^2 f(x)\, dx - I^2\right].$$

3. The PDF of a random variable in $\mathbb{R}$ is given by
$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \leq x \leq 1; \\ 2 - x & \text{if } 1 < x \leq 2; \\ 0 & \text{if } x > 2. \end{cases}$$
   (1) Use the inversion method to derive a formula for generating a random sample $X$ according to $f$ using a random number $U \sim \mathcal{U}[0,1]$. Then, generate $N = 1,000$ samples. Plot the histogram of these $N$ samples on top of the graph of $f$ with 100 bins.
   (2) Use the acceptance-rejection method with the proposal density $g(x) = 1/2$ for $0 \leq x \leq 2$ and $g(x) = 0$ otherwise to sample $N = 1,000$ random variables according to $f$. Plot the histogram of these $N$ samples on top of the graph of $f$ with 100 bins.

4. Generate $N = 100$ sample points $(X_i, Y_i)$ $(i = 1, \ldots, N)$ tht are uniformly distributed inside the ellipse $5x^2 - 4xy + 5y^2 = 48$ using the change-of-coordinates method. Plot these sample points.

5. Generate $N = 1,000$ points uniformly on the unit circle $x^2 + y^2 = 1$. Calculate the sample mean and sample covariance matrix based on these sample points.