Math 114/214: Introduction to Computational Stochastics
Spring quarter 2025

Homework Assignment 1

Due: 3:00 pm, Wednesday, April 9, 2025

Instructions. Please submit your solution as a single PDF file to Gradescope. You are encouraged to type out your solution. If you write your solution, please write it neatly, and then scan it into a PDF file. Using cell phone cameras is discouraged due to low resolution.

1. Write a code to generate points (x_i, y_i) $(i = 1, \dots, N)$ uniformly and independently on $[-1, 1]$, count the total number $\tau(N)$ of points falling into the unit circle (i.e., those points (x_i, y_i) such that $x_i^2 + y_i^2 \le 1$), and calculate $\pi_N = 4\tau(N)/N$ and $e_N := |\pi_N - \pi|$ for $N = 100, 1,000$, $10,000$, and $100,000$. For the case $N = 100$, plot the unit circle and the square (bound by $x = \pm 1$ and $y = \pm 1$), and continue to plot those points inside the unit circle using one color and those points outside the unit circle using a different color. Your solution should include a code, a table showing N, π_N, and e_N, and a plot.

2. Use the Monte Carlo method to calculate approximate values of the integral $I = \int_{[0,1]^d} g(x)\, dx$, where $d = 4$ and $g(x) = \prod_{i=1}^{4} x_i e^{-x_i^2/2}$.
   (1) Evaluate the value I of the integral directly.
   (2) Write a code to generate independent sample points $X_1, X_2, \dots X_N$ that are uniformly distributed in $[0, 1]^4$, and calculate $I_N = (1/N) \sum_{k=1}^{N} g(X_k)$ and $e_N := |I_N - I|$ for $N = 10^2, 10^3, 10^4, 10^5, 10^6$.
   (3) Create a table to show N, I_N, and e_N, and plot e_N vs. N in the log-log scale.
   (4) Discuss your results.

   Your solution should include a code, a table showing N, I_N, and e_N, and a plot.

3. Let $\lambda > 0$. Recall that the cumulative distribution function for a random variable X that is exponentially distributed with the parameter $\lambda > 0$ is given by $F(x) = 0$ if $x < 0$ and $F(x) = 1 - e^{-\lambda x}$ if $x \ge 0$. Derive a formula for sampling the random variable X using $U[0, 1]$.

4. The Box–Muller method for sampling the two-dimensional random variables from the Gaussian distribution
$$g(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2},$$
   using the transformation $x = r\cos\theta$ and $y = r\sin\theta$. The distribution for r is then given by $\alpha(r) = re^{-r^2/2}$ for $r > 0$ and $\alpha(r) = 0$ if $r \le 0$, and that for θ is θ is uniform on $[0, 2\pi]$. To sample $(X_1, Y_1), \dots, (X_N, Y_N)$ from g, one can first sample $R_1, \dots, R_N$ from α using the inversion method and sample $\Theta_1, \dots, \Theta_N$ from the uniform distribution on $[0, 2\pi]$. Then, one can obtain the samples (X_i, Y_i) by the relations $X_i = R_i \cos\Theta_i$ and $Y_i = R_i \sin\Theta_i$.

   Sample $(X_1, Y_1), \dots, (X_N, Y_N)$ from g with $N = 100, 200, 300, 400, 500$. For each of these N-values, calculate the sample mean and the sample covariance matrix, and compare them with the mean and covariance of a g-distributed random variable. Your solution should include a pseudo code and the sample means and covariance matrices but not all the sample values.