Math 105: Homework 4

Due October 31, 2025

Most questions are from the textbook but have been copied here for your convenience.

1. In 1835, Gauss gave the following construction for writing a prime congruent to $1 \pmod 4$ as the sum of two squares: Let $p = 4k+1$ be a prime number. Determine x (this is uniquely possible by Theorem 3.4 in [Stark]) so that
$$x \equiv \frac{(2k)!}{2(k!)^2} \pmod p, |x| < \frac{p}{2}.$$
Now determine y so that
$$y \equiv x \cdot (2k)! \pmod p, |y| < \frac{p}{2}.$$
Gauss showed that $x^2+y^2 = p$. Verify Gauss's result for $p = 5$ and $p = 13$.
2. Show that for infinitely many n, the integer 43 divides $n^2 + n + 41$.
3. For an integer $n > 1$, show that if $n^2 + 2$ is prime, then $3 \mid n$.
4. Solve $3x \equiv 2 \pmod{78}$
5. Solve $9x \equiv 21 \pmod{12}$.
6. Solve for x that simultaneously satisfies $x \equiv 7 \pmod 9$, $x \equiv 13 \pmod{23}$, and $x \equiv 1 \pmod 2$.
7. Solve $5x + 4y \equiv 6 \pmod 7$ and $3x - 2y \equiv 6 \pmod 7$ simultaneously.

As a reminder, please write clearly and fully explain your solutions. It is OK (and even encouraged) to work with your classmates to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. You are not allowed to search the internet and/or use LLMs to aid you in completing this homework.