Math 104B Homework 1

Winter 2025

This homework is due on gradescope Friday, January 17th at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Asymptotic Notation, 25 points)**.** *Prove the following facts about our asymptotic notation:*

(a) If A and B are both positive, then $A + B = \Theta(\max(A, B))$. [5 points]

(b) For c any positive constant $c \cdot X = \Theta(X)$. [5 points]

(c) For any polynomial $p(x)$ of degree-d, we have that for $|x| \geq 1$, $p(x) = O(|x|^d)$. Furthermore, if the leading term is ax^d, $p(x) \sim ax^d$. [5 points]

(d) For any constants $A, a > 0$, and $x > 2$, $\log^A(x) = o(x^a)$ and $x^A = o(\exp(x^a))$. [5 points]

(e) If $f(n) = O(g(n))$, and $g(n) \geq 0$ then

$$\sum_{n \leq x} f(n) = O\left(\sum_{n \leq x} g(n)\right).$$

[5 points]

Question 2 (Expansion of ζ, 20 points)**.**

(a) By approximating the sum by an integral, prove that for $s > 1$

$$\zeta(s) = \frac{1}{s-1} + C + O(s-1)$$

for some constant C. [10 points]

(b) By approximating the first N terms of the same sum by an integral for $s = 1$, prove that C is the Euler-Mascheroni constant γ. [10 points]

Question 3 (Average Size of the Divisor Function in Arithmetic Progressions, 25 points)**.** *For m a positive integer show that there is a constant C_m so that for $x > 2$,*

$$\sum_{n \leq x, n \equiv 1 \pmod{m}} d(n) = \frac{\phi(m) x \log(x)}{m^2} + C_m x + O_m(\sqrt{x}).$$

Extra Credit [10 points]: evaluate C_m in terms of a sum over divisors of m and the Euler-Mascheroni constant.

Hint: you may need to approximate

$$\sum_{n \leq x, (n,m)=1} \frac{1}{n}.$$

This is the same as

$$\sum_{n \leq x} \frac{1}{n} \sum_{d | (n,m)} \mu(d).$$

Then try interchanging the order of summation.

Question 4 (Deficient and Super-Deficient Numbers, 30 points)**.** *A number n is called perfect if $\sigma(n) = 2n$ and deficient if $\sigma(n) < 2n$.*

(a) *Show that at least a constant fraction of the numbers between 1 and N are deficient. Hint: consider the average size of $\sigma(n)$. [10 points]*

(b) *Show that in fact for any $\epsilon > 0$ that a constant fraction of the numbers between 1 and N (where the fraction depends on ϵ but not N) have $\sigma(n) \leq (1+\epsilon)n$. Hint: You may want to focus your attention on n in some arithmetic progression. [20 points]*

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*