Homework due Friday, October 18, 11:59pm

(1) Let $\omega = e^{2\pi i/3} = \dfrac{-1+i\sqrt{3}}{2}$ Prove that the ring $\mathbb{Z}[\omega] := \{a+b\omega : a, b \in \mathbb{Z}\}$ is a Euclidean domain. Hint: use the valuation function $s(a+b\omega) = |a+b\omega|^2 = a^2 - ab + b^2$.
(2) Show that the polynomial ring $\mathbb{Z}[x]$ is *not* a Euclidean domain. (Hint: find an ideal that is not a principal ideal)
(3) Let R be a commutative ring. Prove $R[x]$ is a PID if and only if R is a field.
(4) (a) Show that if $\gcd(b, c) = 1$, then $\gcd(a, bc) = \gcd(a, b)\gcd(a, c)$.
   (b) Show that for all $x, y \in \mathbb{Z}$, we have $\gcd(bx+cy, bc) = \gcd(b, y)\gcd(c, x)$.
   (c) Let m and n be relatively prime. Show that $\gcd(mx + ny, mn) = 1$ if and only if $\gcd(x, n) = \gcd(y, m) = 1$.
(5) Show that the sequence given by $a_n = n^n$ $(n \geq 1)$ is periodic modulo 3. That is, there exists some t such that $a_{n+t} \equiv a_n \pmod 3$ for all n.
(6) Show that there are no points $(x, y) \in \mathbb{Q} \times \mathbb{Q}$ on the circle $x^2 + y^2 = 3$.

Additional Problems not to be turned in:

(1) Show that $\mathbb{Z}_n$ is a field if and only if n is prime.
(2) Show that if $\{a_1, \ldots, a_m\}$ is a complete residue system modulo m for $m > 2$, then $\{a_1^2, \ldots, a_m^2\}$ is not.
(3) Show that $\phi(n) \to \infty$ as $n \to \infty$.
(4) Show that if $m \mid n$, then $\phi(m) \mid \phi(n)$.