Homework due Friday, October 11, 11:59pm

(1) An *ordered ring* is a (commutative) ring R with a total order $\leq$ (*total* order just means any two elements can be compared) such that for all $a, b, c \in R$,
- If $a \leq b$, then $a + c \leq b + c$
- If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

  (a) Show that if $a < 0$, then a is not a square. That is, there is no b such that $a = b^2$.
  (b) Show that $\mathbb{C}$ is *not* an ordered ring.

(2) Show that if a and b are relatively prime, then $\gcd(a - b, a + b) \in \{1, 2\}$.

(3) Prove or disprove the following implications:
  (a) $\gcd(a, b) = \gcd(a, c) \implies \gcd(a^2, b^2) = \gcd(a^2, c^2)$
  (b) $\gcd(a, b) = \gcd(a, c) \implies \gcd(a, b) = \gcd(a, b, c)$
  (c) Let p be a prime. $(p \mid a^2 + b^2 \wedge p \mid b^2 + c^2) \implies p \mid a^2 + c^2$

(4) Prove that a positive odd integer n has a unique representation $n = x^2 - y^2$ if and only if n is prime.

(5) Show that $(3 + \sqrt{10})^n$ is a unit in $\mathbb{Z}[\sqrt{10}] := \{a + b\sqrt{10} : a, b \in \mathbb{Z}\}$ for all n.

(6) Show that a Euclidean domain R is a *principal ideal domain*, that is every ideal I is of the form $I = (a) = \{ra : r \in R\}$.

Additional Problems not to be turned in: